Local risk-minimization for multidimensional Lévy markets

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Introduction

Hedging problem

- Risky assets (e.g. stock): $S = (S^1, \cdots, S^d)^T, d \geq 1$
- A risk-less asset (e.g. cash, bond): B
- Financial product: $F = F(t, S_t)$ (e.g. $(S^i_T - K)^+$, $(\sum_{i=1}^d l_i S^i_T - K)^+, l_i \in \mathbb{R}, K > 0$, l_i depend on S^i, $(S^1_T - S^2_T)^+, d = 2$)

A investor sells an option and wants to replicate its payoff $F(T, S_T)$ by trading in stocks (liquid assets).

- $\xi_t = (\xi^1_t, \cdots, \xi^d_t)^T$ and η_t: the amount of units of the risky assets and the risk-free asset an investor holds at time t
- The market value of the portfolio at time t: $V_t = \xi_t \cdot S_t + \eta_t B_t$

Hedging strategy $\varphi = (\xi, \eta)$

Investment in risky assets and cash in order to reduce the risk related to a financial product.

Complete market

- perfect replication by self-financing strategies
- martingale representation: $V_t/B_t = V_0 + \int_0^t \tilde{\xi}_u \cdot d\tilde{S}_u$ and $F = V_0 + \int_0^T \tilde{\xi}_t \cdot d\tilde{S}_t$, where $\tilde{S}_t = \frac{S_t}{B_t}$.
- the claim can be replicated at time T with initial investment V_0 and the following strategy at time t :

$$\varphi_t = (\tilde{\xi}_t, V_0 + \int_0^t \tilde{\xi}_u \cdot d\tilde{S}_u - \tilde{\xi} \cdot \tilde{S}_u).$$

Black-Scholes model

- Stock: $dS_t = \sigma S_t dW_t + \mu S_t dt, S_0 > 0, \mu \in \mathbb{R}, \sigma > 0$
- Bond: $B_t = e^{rt}$

Incomplete market

However, it is said that the real market is incomplete in general.

- jumps, stochastic volatility or trading constraints
- martingale representation above does not hold
- ‘every claim attainable and replicated by self-financing strategy’ is not valid.

Hence, we have to choose a suitable hedging method for incomplete market model. We present in this talk (locally) risk-minimizing that is a well-known hedging method for contingent claims in a quadratic way for incomplete financial markets.

- Mean-variance hedging: $\min \mathbb{E}[|\tilde{V}_T - F|^2]$, φ: self-financing strategies
- Risk-minimizing hedging: $\min \mathbb{E}[(C_T - C_t)^2|\mathcal{F}_t]$, φ: mean self-financing strategies with $\tilde{V}_T = F$.

LRM

- Locally risk-minimizing (LRM, for short) is a well-known hedging method for contingent claims in a quadratic way for imcomplete markets.
- Theoretical aspects of LRM have been developed to a high degree. (Contributor: Föllmer, Schweizer, Sondermann and many others)
- But the theory does not give its explicit representation.
- Arai and Suzuki obtained a formula of locally risk-minimizing for Lévy markets under many additional conditions by using Malliavin calculus for Lévy processes.
- In this talk, we obtain an explicit representation of LRM in an incomplete financial market driven by a multidimensional Lévy process by using Malliavin calculus because in real markets, investors sell an option and want to replicate its payoff $F(T, S_T)$ by trading many stocks (liquid assets).

Preliminaries

Model description

We begin with preparation of the probabilistic framework and the underlying Lévy process X under which we discuss Malliavin calculus in the sequel (see e.g., Solé et al. (2007) [1]).

- $T > 0$: a finite time horizon
- $(\Omega_W, \mathcal{F}_W, \mathbb{P}_W)$: a one-dimensional Wiener space on $[0, T]$; and W a one-dimensional standard Brownian motion with $W_0 = 0$.
- $(\Omega_J, \mathcal{F}_J, \mathbb{P}_J)$: the canonical Lévy space for a pure jump Lévy process J on $[0, T]$ with Lévy measure ν_0, that is, $\Omega_J = \cup_{n=0}^{\infty}([0, T] \times \mathbb{R}_0)^n$, where $\mathbb{R}_0 := \mathbb{R} \setminus \{0\}$; and $J_t(\omega_J) = \sum_{i=1}^n z_i 1_{\{t_i \le t\}}$ for $t \in [0, T]$ and $\omega_J = ((t_1, z_1), \ldots, (t_n, z_n)) \in ([0, T] \times \mathbb{R}_0)^n$. Note that $([0, T] \times \mathbb{R}_0)^0$ represents an empty sequence.
- We assume that $\int_{\mathbb{R}_0} z^2 \nu_0(dz) < \infty$.
- We denote $(\Omega^0, \mathcal{F}^0, \mathbb{P}^0) = (\Omega_W \times \Omega_J, \mathcal{F}_W \times \mathcal{F}_J, \mathbb{P}_W \times \mathbb{P}_J)$.
- $\mathbb{F}^0 = \{\mathcal{F}_t^0\}_{t \in [0,T]}$: the canonical filtration completed for $\mathbb{P}^0$.

[1] J. L. Solé, F. Utzet, J. Vives, Canonical Lévy process and Malliavin calculus, Stochastic Process. Appl. 117 (2007) 165–187.

Model description

- X^0: a square integrable centered Lévy process on $(\Omega^0, \mathcal{F}^0, \mathbb{P}^0)$ represented as

$$X_t^0 = \sigma_0 W_t + J_t - t\int_{\mathbb{R}_0} z\nu_0(dz), \tag{1}$$

  where $\sigma_0 > 0$.

Denoting by N the Poisson random measure defined as $N(t, A) := \sum_{s\le t} \mathbf{1}_A(\Delta X_s^0)$, $A \in \mathcal{B}(\mathbb{R}_0)$ and $t \in [0, T]$, where $\Delta X_s^0 := X_s^0 - X_{s-}^0$, we have $J_t = \int_0^t \int_{\mathbb{R}_0} zN(ds, dz)$. In addition, we define its compensated measure as $\widetilde{N}(dt, dz) := N(dt, dz) - \nu_0(dz)dt$. Thus, we can rewrite (1) as

$$X_t^0 = \sigma_0 W_t + \int_0^t \int_{\mathbb{R}_0} z\widetilde{N}(ds, dz). \tag{2}$$

Model description

Now, let $(\Omega^1, \mathcal{F}^1, \mathbb{P}^1), \cdots, (\Omega^d, \mathcal{F}^d, \mathbb{P}^d)$ be d independent copies of $(\Omega^0, \mathcal{F}^0, \mathbb{P}^0)$ for some $d \geq 1$. We set $(\Omega, \mathcal{F}, \mathbb{P}) = (\Omega_1 \times \cdots \times \Omega_d, \mathcal{F}_1 \times \cdots \times \mathcal{F}_d, \mathbb{P}_1 \times \cdots \times \mathbb{P}_d)$ and we call it multidimensional canonical space. Let $X = (X^1, \cdots, X^d)$ be a d-dimensional square integrable centered Lévy process on $(\Omega, \mathcal{F}, \mathbb{P})$ where

$$X_t^j = \sigma_j W_{j,t} + \int_0^t \int_{\mathbb{R}_0} z\widetilde{N}_j(ds, dz), 1 \leq j \leq d$$

where $\sigma_j > 0, W_{j,t}$ a Brownian motion on $(\Omega^j, \mathcal{F}^j, \mathbb{P}^j)$, $\tilde{N}_j$ the compensated Poisson random measure on $(\Omega^j, \mathcal{F}^j, \mathbb{P}^j)$ has Lévy measure ν_j satisfies $\int_{\mathbb{R}_0} z^2 \nu_j(dz) < \infty$.

Model description

Market model

We consider a financial market being composed of one riskfree asset and d *risky assets with finite time horizon* $T > 0$*:*
Riskfree asset price process:

$$B_t = 1, t \in [0, T],$$

Risky assets price processes:

$$dS_t^i = S_{t-}^i \left[\alpha_t^i dt + \beta_{i,t} dW_{i,t} + \int_{\mathbb{R}_0} \gamma_{i,t,z} \widetilde{N}_i(dt, dz) \right], \quad S_0^i > 0, i = 1, \cdots, d \quad (3)$$

where α*,* β *and* γ *are predictable processes satisfying the following:*

Model description

Assumption (A)

1. *(3) has a solution* $\boldsymbol{S}$ *satisfying the so-called structure condition (SC, for short). That is,* $\boldsymbol{S}$ *is a special semimartingale with the canonical decomposition* $\boldsymbol{S} = \boldsymbol{S}_0 + \boldsymbol{M} + \boldsymbol{A}$ *such that*

$$\sum_{i=1}^{d} \left\| [M^i]_T^{1/2} + \int_0^T |dA_s^i| \right\|_{L^2(\mathbb{P})} < \infty, \tag{4}$$

where $\boldsymbol{M} = (M^1, \cdots, M^d)^T$, $\boldsymbol{A} = (A^1, \cdots, A^d)^T$, $dM_t^i = S_{t-}^i(\beta_{i,t} dW_{i,t} + \int_{\mathbb{R}_0} \gamma_{i,t,z} \widetilde{N}_i(dt, dz))$ *and* $dA_t^i = S_{t-}^i \alpha_t^i dt$ *for* $i = 1, \cdots, d$*. Moreover, defining a process*

$$\lambda_t^i := \frac{\alpha_t^i}{S_{t-}^i(\beta_{i,t}^2 + \int_{\mathbb{R}_0} \gamma_{i,t,z}^2 \nu_i(dz))},$$

we have $A^i = \int \lambda d\langle M^i \rangle$*. Thirdly, the mean-variance trade-off process* $K_t^i := \int_0^t \lambda_s^2 d\langle M^i \rangle_s$ *is finite, that is,* K_T^i *is finite* $\mathbb{P}$*-a.s.*

2. $\gamma_{i,t,z} > -1$, (t, z, ω)*-a.e. for* $i = 1, \cdots, d$*, that is,* $\mathbb{E}\left[\int_0^T \int_{\mathbb{R}_0} \mathbf{1}_{\{\gamma_{i,t,z} \leq -1\}} \nu_i(dz) dt\right] = 0.$

Definition of locally risk-minimizing

Definition 2.1

1. $\Theta_S := \{\xi = (\xi^1, \cdots, \xi^d)^T : \mathbb{R}^d$ **– valued predictable process |** $\mathbb{E}\left[\sum_{i=1}^d \int_0^T (\xi_t^i)^2 d\langle M^i\rangle_t + (\sum_{i=1}^d \int_0^T |\xi_t^i dA_t^i|)^2\right] < \infty\}.$
2. An L^2-strategy is given by a pair $\varphi = (\xi, \eta)$, where $\xi \in \Theta_S$ and η is an adapted process such that $V(\varphi) := \xi \cdot S + \eta = \sum_{i=1}^d (\xi^i) S^i + \eta$ is a right continuous process with $\mathbb{E}[V_t^2(\varphi)] < \infty$ for every $t \in [0, T]$. Note that ξ_t (resp. η_t) represents the amount of units of the risky assets (resp. the riskfree asset) an investor holds at time t.
3. For $F \in L^2(\mathbb{P})$, the process $C^F(\varphi)$ defined by $C_t^F(\varphi) := F 1_{\{t=T\}} + V_t(\varphi) - \sum_{i=1}^d \int_0^t \xi_s^i dS_s^i$ is called the cost process of $\varphi = (\xi, \eta)$ for F.
4. An L^2-strategy φ is said locally risk-minimizing for F if $V_T(\varphi) = 0$ and $C^F(\varphi)$ is a martingale orthogonal to each M^i, $1 \leq i \leq d$, that is, $[C^F(\varphi), M^i] (1 \leq i \leq d)$ is a uniformly integrable martingale.

· If φ is self-financing, then $C(\varphi)$ is a constant.
If there exists a self-financing φ s.t. $V_T(\varphi) = 0$, we have
$F = V_0(\varphi) + \int_0^T \xi_s \cdot dS_s$. This is a contradiction!

· An L^2-strategy φ^* for F is risk-minimizing if $V_T(\varphi^*) = 0$ and $R_t(\varphi^*) \leq R_t(\varphi), \forall t \in [0, T]$, hold for all φ such that $V_T(\varphi) = 0$, where
$R_t(\varphi) := \mathbb{E}[(C_T(\varphi) - C_t(\varphi))^2 | \mathcal{F}_t]$.

Föllmer-Schweizer decomposition I

- In order to obtain a representation of LRM, Föllmer-Schweizer decomposition (FS decomposition, for short) can be very useful.

Definition 2.2

An $F \in L^2(\mathbb{P})$ admits a Föllmer-Schweizer decomposition if it can be described by

$$F = F_0 + \int_0^T \xi_t^F \cdot dS_t + L_T^F,$$

where $F_0 \in \mathbb{R}$, $\xi^F \in \Theta_S$ and L^F is a square-integrable martingale orthogonal to M with $L_0^F = 0$.

Proposition 2.3 (Proposition 5.2 of Schweizer 2008.[a])

[a] M. Schweizer, Local Risk-Minimization for Multidimensional Assets and Payment Streams, Banach Center Publ. 83 (2008) 213–229.

An LRM $\varphi = (\xi, \eta)$ for F exists if and only if F admits an FS decomposition, and its relationship is given by

$$\xi_t = \xi_t^F, \quad \eta_t = F_0 + \int_0^t \xi_s^F \cdot dS_s + L_t^F - F\mathbf{1}_{\{t=T\}} - \xi_t^F \cdot S_t.$$

Föllmer-Schweizer decomposition II

We next denote $Z := \mathcal{E}(-\int \lambda \cdot dM)$, where $\mathcal{E}(Y)$ represents the stochastic exponential of Y, that is, Z is a solution of the SDE $dZ_t = -\sum_{i=1}^{d} \lambda_t^i Z_{t-} dM_t^i$. In addition to Assumption (A), we suppose the following:

Assumption (B)

Z is a positive square integrable martingale; and $Z_T F \in L^2(\mathbb{P})$.

Definition 2.4 (Minimal martingale measure)

A martingale measure $\mathbb{P}^* \sim \mathbb{P}$ is called minimal if any square-integrable $\mathbb{P}$-martingale orthogonal to M remains a martingale under $\mathbb{P}^*$.

Proposition 2.5

Under Assumption (A), if Z is a positive square integrable martingale, then a minimal martingale measure $\mathbb{P}^*$ exists with $d\mathbb{P}^* = Z_T d\mathbb{P}$.

Föllmer-Schweizer decomposition III

Under Assumptions (A) and (B), we discuss a representation of ξ^F. If Z is a positive square integrable martingale; and $Z_T F \in L^2(\mathbb{P})$, then, The martingale representation theorem (see, e.g. section 2 of Benth et al. [?]) provides

$$Z_T F = \mathbb{E}_{\mathbb{P}^*}[F] + \sum_{i=1}^{d} \int_0^T g_t^{i,0} dW_{i,t} + \sum_{i=1}^{d} \int_0^T \int_{\mathbb{R}_0} g_{t,z}^{i,1} \widetilde{N}_i(dt, dz)$$

for some predictable processes $g_t^{i,0}$ and $g_{t,z}^{i,1}$, $1 \leq i \leq d$. By the same sort of calculations as the proof of Theorem 4.4 in Suzuki (2013), we have

$$\begin{aligned} F &= \mathbb{E}_{\mathbb{P}^*}[F] + \sum_{i=1}^{d} \int_0^T \frac{g_t^{i,0} + \mathbb{E}[Z_T F | \mathcal{F}_{t-}] u_{i,t}}{Z_{t-}} dW_{i,t}^{\mathbb{P}^*} \\ &\quad + \sum_{i=1}^{d} \int_0^T \int_{\mathbb{R}_0} \frac{g_{t,z}^{i,1} + \mathbb{E}[Z_T F | \mathcal{F}_{t-}] \theta_{i,t,z}}{Z_{t-}(1 - \theta_{i,t,z})} \widetilde{N}_i^{\mathbb{P}^*}(dt, dz) \\ &=: \mathbb{E}_{\mathbb{P}^*}[F] + \sum_{i=1}^{d} \int_0^T h_t^{i,0} dW_{i,t}^{\mathbb{P}^*} + \sum_{i=1}^{d} \int_0^T \int_{\mathbb{R}_0} h_{t,z}^{i,1} \widetilde{N}_i^{\mathbb{P}^*}(dt, dz) \end{aligned}$$

where $u_{i,t} := \lambda_t^i S_{t-}^i \beta_{i,t}$, $\theta_{i,t,z} := \lambda_t^i S_{t-}^i \gamma_{i,t,z}$, $dW_{i,t}^{\mathbb{P}^*} := dW_{i,t} + u_{i,t} dt$ and $\widetilde{N}^{\mathbb{P}^*_i}(dt, dz) := \widetilde{N}_i(dt, dz) + \theta_{i,t,z} \nu_i(dz) dt$. Girsanov's theorem implies that $W_i^{\mathbb{P}^*}$ and $\widetilde{N}_i^{\mathbb{P}^*}$ are Brownian motions and the compensated Poisson random measures of N_i under $\mathbb{P}^*$, respectively.

Föllmer-Schweizer decomposition IV

Additionally, we assume that

$$\sum_{i=1}^{d} \mathbb{E}\left[\int_0^T \left\{(h_t^{i,0})^2 + \int_{\mathbb{R}_0} (h_{t,z}^{i,1})^2 \nu_i(dz)\right\} dt\right] < \infty. \tag{5}$$

Denoting $i_t^{j,0} := h_t^{j,0} - \xi_t^i S_{t-}^i \beta_{i,t}$, $i_{t,z}^{i,1} := h_{t,z}^{i,1} - \xi_t^i S_{t-}^i \gamma_{i,t,z}$, and

$$\xi_t^i := \frac{\lambda_t^i}{\alpha_t^i}\{h_t^{i,0}\beta_{i,t} + \int_{\mathbb{R}_0} h_{t,z}^{i,1}\gamma_{i,t,z}\nu_i(dz))\}, \tag{6}$$

we can see

$$i_t^{i,0}\beta_{i,t} + \int_{\mathbb{R}_0} i_{t,z}^{j,1}\gamma_{i,t,z}\nu_i(dz) = 0$$

for any $t \in [0, T]$, which implies $i_t^{i,0}u_{i,t} + \int_{\mathbb{R}_0} i_{t,z}^{i,1}\theta_{i,t,z}\nu_i(dz) = 0$. We have then

$$\begin{aligned} F - \mathbb{E}_{\mathbb{P}^*}[F] - \int_0^T \xi_t dS_t &= \sum_{i=1}^{d}\int_0^T i_t^{i,0} dW_{i,t}^{\mathbb{P}^*} + \sum_{i=1}^{d}\int_0^T \int_{\mathbb{R}_0} i_{t,z}^{i,1}\widetilde{N}_i^{\mathbb{P}^*}(dt, dz) \\ &= \sum_{i=1}^{d}\int_0^T i_t^{i,0} dW_{i,t} + \sum_{i=1}^{d}\int_0^T \int_{\mathbb{R}_0} i_{t,z}^{i,1}\widetilde{N}_i(dt, dz). \end{aligned}$$

Föllmer-Schweizer decomposition IV

Now we denote

$$\xi_t^i := \frac{\lambda_t^i}{\alpha_t^i}\{h_t^{i,0}\beta_{i,t} + \int_{\mathbb{R}_0} h_{t,z}^{i,1}\gamma_{i,t,z}\nu_i(dz))\} \tag{1.1}$$

Under this setting, we can derive the following:

Theorem 2.6

Assuming that Assumptions (A), (B) and $\sum_{i=1}^d \mathbb{E}\left[\int_0^T \left\{(h_t^{i,0})^2 + \int_{\mathbb{R}_0}(h_{t,z}^{i,1})^2\nu_i(dz)\right\}dt\right] < \infty$, we have $\xi^F = \xi$ defined in (1.1).

Remark 2.7

The processes h^0 and h^1 appeared in (1.1) is implied by the martingale representation theorem. Thus, it is almost impossible to calculate them explicitly. We next reduce an explicit representation of ξ^F by using Malliavin calculus for Lévy processes.

Reminder: $\lambda_t^i := \dfrac{\alpha_t^i}{S_{t-}^i(\beta_{i,t}^2 + \int_{\mathbb{R}_0}\gamma_{i,t,z}^2\nu_i(dz))}$,

$$dS_t^i = S_{t-}^i\left[\alpha_t^i dt + \beta_{i,t}dW_{i,t} + \int_{\mathbb{R}_0}\gamma_{i,t,z}\widetilde{N}_i(dt,dz)\right], \quad S_0^i > 0, i = 1,\cdots,d.$$

Main theorem

Multiple stochastic integral

Definition 3.1

- $q^j(E) = \sigma_j^2 \int_E dt\delta_0(dz) + \int_E z^2 dt\nu_j(dz), \quad E \in \mathcal{B}([0,T] \times \mathbb{R}),$
  $E \in \mathcal{B}([0,T] \times \mathbb{R})$
- $Q_j(E) = \sigma_j \int_E dW_{j,t}\delta_0(dz) + \int_E z\tilde{N}_j(dt,dz)$
- We consider the product of the form

$$\mathbb{H}_\alpha(\omega) := \prod_{j=1}^{d} I_{\alpha^{(j)}}(f_{j,\alpha^{(j)}})(\omega_j)$$

for any $\alpha \in \mathcal{J}^d$, which is the set of indexes of the form $\alpha = (\alpha^{(1)}, \cdots, \alpha^{(d)})$ with $\alpha^{(j)} = 0, 1, \cdots$, for $j = 1, \cdots, d$. Here $I_{\alpha^{(j)}}(f_{j,\alpha^{(j)}})$ is the $\alpha^{(j)}$-fold iterated Itô integral with respect to random measure Q and $f_{j,\alpha^{(j)}}$ is deterministic function satisfying

$$\int_{([0,T]\times\mathbb{R})^{\alpha^{(j)}}} |f_{j,\alpha^{(j)}}((t_1,z_1),\cdots,(t_{\alpha^{(j)}},z_{\alpha^{(j)}}))|^2 q^j(dt_1,dz_1)\cdots q^j(dt_{\alpha^{(j)}},dz_{\alpha^{(j)}}) < \infty.$$

Malliavin derivative

The elements $\mathbb{H}_\alpha, \alpha \in \mathcal{J}^d$, constitute an orthogonal basis in $L^2(\mathbb{P})$. Any real $\mathcal{F}_T$ -measurable random variable $F \in L^2(\mathbb{P})$ can be written as

$$F = \sum_{\alpha \in \mathcal{J}^d} \mathbb{H}_\alpha$$

for an appropriate choice of deterministic symmetric integrands in the iterated Itô integrals.

Definition 3.2

(1) Let $\mathbb{D}^{1,2}$ denote the set of $\mathcal{F}$ -measurable random variables $F \in L^2(\mathbb{P})$ with the representation $F = \sum_{\alpha \in \mathcal{J}^d} \mathbb{H}_\alpha, \prod_{j=1}^d I_{\alpha^{(j)}}(f_{j,\alpha^{(j)}})(\omega_j)$ satisfying $\sum_{j=1}^d \sum_{\alpha \in \mathcal{J}^d} \alpha^{(j)} \alpha^{(j)}! \| f_{j,\alpha^{(j)}} \|^2_{L^2(([0,T]\times\mathbb{R})^{\alpha^{(j)}})} < \infty$.

(2) Let $F \in \mathbb{D}^{1,2}$. Then we define the Malliavin derivative DF of a random variable $F \in \mathbb{D}^{1,2}$ as the gradient $D_{t,z}F = (D^1_{t,z}F, \cdots, D^d_{t,z}F)$ where

$$D^j_{t,z}F := \sum_{\alpha \in \mathcal{J}^d} \alpha^{(j)} \mathbb{H}_{\alpha - \epsilon^{(j)}}(t,z), t \in [0,T], z \in \mathbb{R}, j = 1, \cdots, d.$$

Here $\epsilon^{(j)} = (0, \cdots, 0, 1, 0, \cdots, 0)$ with 1 in the j-th position.

Clark-Ocone formula

Proposition 3.3 (Clark-Ocone type formula for multidimensional Lévy functionals)

Let $F \in \mathbb{D}^{1,2}$. Then, we have

$$F = \mathbb{E}[F] + \sum_{i=1}^{d} \int_{[0,T]\times\mathbb{R}} \mathbb{E}[D^i_{t,z} F | \mathcal{F}_{t-}] Q_i(dt, dz)$$
$$= \mathbb{E}[F] + \sum_{i=1}^{d} \sigma_i \int_0^T \mathbb{E}[D^i_{t,0} F | \mathcal{F}_{t-}] dW_{i,t} + \sum_{i=1}^{d} \int_0^T \int_{\mathbb{R}_0} \mathbb{E}[D^i_{t,z} F | \mathcal{F}_{t-}] z \tilde{N}_i(dt, dz).$$

Assumption (C)

1. $u_i, u_i^2 \in \mathbb{L}_0^{j,1,2}$; *and* $2u_{i,s}D^j_{t,z}u_{i,s} + z(D^j_{t,z}u_{i,s})^2 \in L^2(q^j \times \mathbb{P})$ *for a.e.* $s \in [0,T], i,j = 1,\cdots,d$.
2. $\theta_i + \log(1-\theta_i) \in \tilde{\mathbb{L}}_1^{j,1,2}$, *and* $\log(1-\theta_i) \in \mathbb{L}_1^{j,1,2}$, $i,j = 1,\cdots,d$
3. *For* q*-a.e.* $(s,x) \in [0,T] \times \mathbb{R}_0$, *there is an* $\varepsilon_{i,s,x} \in (0,1)$ *such that* $\theta_{i,s,x} < 1 - \varepsilon_{i,s,x}, i = 1,\cdots,d$
4. $Z_T \in L^2(\mathbb{P})$; *and* $Z_T\{D^j_{t,0}\log Z_T 1_{\{0\}}(z) + \frac{e^{zD^j_{t,z}\log Z_T}-1}{z}1_{\mathbb{R}_0}(z)\} \in L^2(q^j \times \mathbb{P})$.
5. $F \in \mathbb{D}^{1,2}$ *with* $FZ_T \in L^2(\mathbb{P})$; *and* $Z_T D^j_{t,z}F + FD^j_{t,z}Z_T + zD^j_{t,z}F \cdot D^j_{t,z}Z_T \in L^2(q^j \times \mathbb{P}), j = 1,\cdots,d$.
6. $FH^{j,*}_{t,z}, H^{j,*}_{t,z}D^j_{t,z}F \in L^1(\mathbb{P}^*), (t,z)$ *-a.e. where* $H^{j,*}_{t,z} = \exp(zD^j_{t,z}\log Z_T - \log(1-\theta_{j,t,z}))$

Reminder: $u_{i,t} = \lambda^i_t S^i_{t-}\beta_{i,t}, \theta_{i,t,z} = \lambda^i_t S^i_{t-}\gamma_{i,t,z}, \lambda^i_t = \dfrac{\alpha^i_t}{S^i_{t-}(\beta^2_{i,t} + \int_{\mathbb{R}_0}\gamma^2_{i,t,z}\nu_i(dz))}$

$$dS^i_t = S^i_{t-}\left[\alpha^i_t dt + \beta_{i,t}dW_{i,t} + \int_{\mathbb{R}_0}\gamma_{i,t,z}\tilde{N}_i(dt,dz)\right], \quad S^i_0 > 0.$$

Definition 3.4

For $1 \leq i, j \leq d$, we define the following:

1. Let $\mathbb{L}^{j,1,2}$ denote the space of product measurable and $\mathbb{F}$-adapted processes $G_i : \Omega \times [0,T] \times \mathbb{R} \to \mathbb{R}$ satisfying
   1. $$\mathbb{E}\left[\int_{[0,T]\times\mathbb{R}} |G_{i,s,x}|^2 q^i(ds,dx)\right] < \infty,$$
   2. $G_{i,s,x} \in \mathbb{D}^{j,1,2}$, q^i–a.e. $(s,x) \in [0,T] \times \mathbb{R}$
   3. $$\mathbb{E}\left[\int_{([0,T]\times\mathbb{R})^2} |D^j_{t,z} G_{i,s,x}|^2 q^i(ds,dx) q^j(dt,dz)\right] < \infty.$$
2. $\mathbb{L}_0^{j,1,2}$ denotes the space of $G : [0,T] \times \Omega \to \mathbb{R}$ satisfying
   1. $G_{i,s} \in \mathbb{D}^{j,1,2}$ for a.e. $s \in [0,T]$,
   2. $E\left[\int_{[0,T]} |G_{i,s}|^2 ds\right] < \infty$,
   3. $E\left[\int_{[0,T]\times\mathbb{R}} \int_0^T |D^j_{t,z} G_{i,s}|^2 ds q^j(dt,dz)\right] < \infty$.
3. $\mathbb{L}_1^{j,1,2}$ is defined as the space of $G : [0,T] \times \mathbb{R}_0 \times \Omega \to \mathbb{R}$ such that
   1. $G_{i,s,x} \in \mathbb{D}^{j,1,2}$ for q^i-a.e. $(s,x) \in [0,T] \times \mathbb{R}$,
   2. $E\left[\int_{[0,T]\times\mathbb{R}_0} |G_{i,s,x}|^2 \nu_i(dx) ds\right] < \infty$,
   3. $E\left[\int_{[0,T]\times\mathbb{R}} \int_{[0,T]\times\mathbb{R}_0} |D^j_{t,z} G_{i,s,x}|^2 \nu_i(dx) ds q^j(dt,dz)\right] < \infty$.
4. $\tilde{\mathbb{L}}_1^{j,1,2}$ is defined as the space of $G \in \mathbb{L}^{1,2}$ such that
   1. $E\left[\left(\int_{[0,T]\times\mathbb{R}_0} |G_{i,s,x}| \nu_i(dx) ds\right)^2\right] < \infty$,
   2. $E\left[\int_{[0,T]\times\mathbb{R}} \left(\int_{[0,T]\times\mathbb{R}_0} |D^j_{t,z} G_{i,s,x}| \nu_i(dx) ds\right)^2 q^j(dt,dz)\right] < \infty$.

Theorem 3.5 (Clark-Ocone type formula under change of measure)

Under Assumptions (B) and (C),

$$F = \mathbb{E}_{\mathbb{P}^*}[F] + \sum_{j=1}^{d} \sigma_j \int_0^T \mathbb{E}_{\mathbb{P}^*}\left[D^j_{t,0} F - F K^j_t \middle| \mathcal{F}_{t-} \right] dW^{\mathbb{P}^*}_{j,t}$$
$$+ \sum_{j=1}^{d} \int_0^T \int_{\mathbb{R}_0} \mathbb{E}_{\mathbb{P}^*}[F(H^{j,*}_{t,z} - 1) + z H^{j,*}_{t,z} D^j_{t,z} F | \mathcal{F}_{t-}] \tilde{N}^{\mathbb{P}^*}_j(dt, dz), a.s.$$

holds, where

$$K^j_t = \sum_{i=1}^{d} \int_0^T D^j_{t,0} u_{i,s} dW^{\mathbb{P}^*}_{i,s} + \sum_{i=1}^{d} \int_0^T \int_{\mathbb{R}_0} \frac{D^j_{t,0}\theta_{i,s,x}}{1-\theta_{i,s,x}} \tilde{N}^{\mathbb{P}^*}_i(ds, dx)$$

and

$$H^{j,*}_{t,z} = \exp\Bigg\{ -\sum_{i=1}^{d} \int_0^T z D^j_{t,z} u_{i,s} dW^{\mathbb{P}^*}_{i,s} - \sum_{i=1}^{d} \frac{1}{2} \int_0^T (z D^j_{t,z} u_{i,s})^2 ds$$
$$+ \sum_{i=1}^{d} \int_0^T \int_{\mathbb{R}_0} \left((1-\theta_{i,s,x}) z D^j_{t,z} \log(1-\theta_{i,s,x}) + z D^j_{t,z} \theta_{i,s,x} \right) \nu_i(dx) ds$$
$$+ \sum_{i=1}^{d} \int_0^T \int_{\mathbb{R}_0} z D^j_{t,z} \log(1-\theta_{i,s,x}) \tilde{N}^{\mathbb{P}^*}_i(ds, dx) \Bigg\}.$$

Theorem 3.6

Under Assumptions (A), (B), (C), $\boldsymbol{h}^0$ and $\boldsymbol{h}^1$ are described as

$$h_t^{j,0} = \sigma_j \mathbb{E}_{\mathbb{P}^*}\left[D_{t,0}^j F - F\sum_{i=1}^d \left[\int_0^T D_{t,0}^j u_{i,s} dW_{i,s}^{\mathbb{P}^*} + \int_0^T \int_{\mathbb{R}_0} \frac{D_{t,0}^j \theta_{i,s,x}}{1-\theta_{i,s,x}} \widetilde{N}_i^{\mathbb{P}^*}(ds,dx)\right] \middle| \mathcal{F}_{t-}\right], \tag{2.1}$$

$$h_{t,z}^{j,1} = \mathbb{E}_{\mathbb{P}^*}[F(H_{t,z}^{j,*} - 1) + zH_{t,z}^{j,*} D_{t,z}^j F | \mathcal{F}_{t-}]. \tag{2.2}$$

where

$$\begin{aligned} H_{t,z}^{j,*} = \exp\Bigg\{ &-\sum_{i=1}^d \int_0^T zD_{t,z}^j u_{i,s} dW_{i,s}^{\mathbb{P}^*} - \sum_{i=1}^d \frac{1}{2}\int_0^T (zD_{t,z}^j u_{i,s})^2 ds \\ &+ \sum_{i=1}^d \int_0^T \int_{\mathbb{R}_0} \left((1-\theta_{i,s,x}) zD_{t,z}^j \log(1-\theta_{i,s,x}) + zD_{t,z}^j \theta_{i,s,x}\right) \nu_i(dx) ds \\ &+ \sum_{i=1}^d \int_0^T \int_{\mathbb{R}_0} zD_{t,z}^j \log(1-\theta_{i,s,x}) \tilde{N}_i^{\mathbb{P}^*}(ds,dx) \Bigg\}. \end{aligned}$$

Theorem 4.3 (Cont'd)

Supposing all conditions of Theorem 2.6 additionally, ξ^F is given by substituting (2.1) and (2.2) for $\mathbf{h}^0$ and $\mathbf{h}^1$ in (1.1) respectively.

Reminder: Theorem 2.6

Assuming that Assumptions (A), (B) and $\sum_{i=1}^d \mathbb{E}\left[\int_0^T \left\{(h_t^{i,0})^2 + \int_{\mathbb{R}_0}(h_{t,z}^{i,1})^2\nu_i(dz)\right\} dt\right] < \infty$, we have $\xi^F = \xi$ defined in (1.1).

$$\xi_t^i := \frac{\lambda_t^i}{\alpha_t^i}\{h_t^{i,0}\beta_{i,t} + \int_{\mathbb{R}_0} h_{t,z}^{i,1}\gamma_{i,t,z}\nu_i(dz)\}. \tag{1.1}$$

Corollary 3.7

In the case where α, β, and γ are given by continuous deterministic functions satisfying Assumption (D), if F and $Z_T F \in \mathbb{D}^{1,2}$, then $\xi^i, i = 1, 2, \cdots, d$ is given as

$$\xi_t^i = \frac{\sigma_i \beta_{i,t} \mathbb{E}_{\mathbb{P}^*}[D_{t,0}^i F | \mathcal{F}_{t-}] + \int_{\mathbb{R}_0} \mathbb{E}_{\mathbb{P}^*}[z D_{t,z}^i F | \mathcal{F}_{t-}] \gamma_{i,t,z} \nu_i(dz)}{S_{t-}^i \left(\beta_{i,t}^2 + \int_{\mathbb{R}_0} \gamma_{i,t,z}^2 \nu_i(dz) \right)}.$$

Assumption (D)

1. $\gamma_{i,t,z} > -1$, (t, z, ω)-a.e.
2. $\sup_{t \in [0,T]} (|\alpha_{i,t}| + \beta_{i,t}^2 + \int_{\mathbb{R}_0} \gamma_{i,t,z}^2 \nu(dz)) < C$ for some $C > 0$.
3. There exists an $\varepsilon > 0$ such that

$$\frac{\alpha_{i,t} \gamma_{i,t,z}}{\beta_{i,t}^2 + \int_{\mathbb{R}_0} \gamma_{i,t,z}^2 \nu_i(dz)} < 1 - \varepsilon \quad \text{and} \quad \beta_{i,t}^2 + \int_{\mathbb{R}_0} \gamma_{i,t,z}^2 \nu_i(dz) > \varepsilon, (t, z, \omega)\text{-a.e.}$$

Remark

1. LRM for Lévy markets (one dimensional) has been also discussed in Vandaele and Vanmaele (2008) without Malliavin calculus. They considered the case where all coefficients in (3) are deterministic; and studied LRM for unit-linked life insurance contracts.
2. Benth et al. (2003) also concerned a similar issue by using Malliavin calculus. They however studied minimal variance portfolio which is different from LRM, and considered only the case where the underlying asset price process is a martingale.
3. Yang et al. (2010) derived an explicit representation of LRM for a European call option in the Hull and White model by using the Malliavin calculus in Wiener space. They also give a numerical result of it.
4. Arai and Suzuki (2015) derived explicit representations of LRM for one dimensional Lévy markets. They also calculated its concrete expressions for call options, Asian options and lookback options.

Example

Swap option

In this section, we consider the case $d = 2$. We calculate the Malliavin derivatives of $(F_1 - F_2)^+$ by using the mollifier approximation, where $x^+ = x \vee 0$, $F = (F_1, F_2) \in \mathbb{D}^{1,2}$.

Theorem 4.1

For any $F = (F_1, F_2) \in \mathbb{D}^{1,2}$ q*-a.e.* $(t, z) \in [0, T] \times \mathbb{R}$*, we have* $(F_1 - F_2)^+ \in \mathbb{D}^{1,2}$,

$$D^i_{t,z}(F_1 - F_2)^+$$

$$= 1_{\{F_1 > F_2\}} D^i_{t,0}(F_1 - F_2) \cdot 1_{\{0\}}(z) + \frac{(F_1 - F_2 + zD^i_{t,z}(F_1 - F_2))^+ - (F_1 - F_2)^+}{z} 1_{\mathbb{R}_0}(z).$$

The deterministic coefficients case

We consider the case where α, β, and γ are continuous deterministic functions satisfying Assumption (D).

Reminder: Assumption (D)

1. $\gamma_{i,t,z} > -1$, (t, z, ω)-a.e.
2. $\sup_{t\in[0,T]}(|\alpha_{i,t}| + \beta^2_{i,t} + \int_{\mathbb{R}_0} \gamma^2_{i,t,z}\nu(dz)) < C$ for some $C > 0$.
3. There exists an $\varepsilon > 0$ such that

$$\frac{\alpha_{i,t}\gamma_{i,t,z}}{\beta^2_{i,t} + \int_{\mathbb{R}_0} \gamma^2_{i,t,z}\nu_i(dz)} < 1 - \varepsilon \quad \text{and} \quad \beta^2_{i,t} + \int_{\mathbb{R}_0} \gamma^2_{i,t,z}\nu_i(dz) > \varepsilon, (t, z, \omega)\text{-a.e.}$$

Moreover, we assume

$$\int_{\mathbb{R}_0} \{\gamma^4_{i,t,z} + |\log(1 + \gamma_{i,t,z})|^2\}\nu_i(dz) < C \text{ for some } C > 0. \tag{4.1}$$

First of all, we calculate the Malliavin derivatives of S_T^i for such cases.

Proposition 4.2

$D_{t,z}^i S_T^i = \frac{S_T^i \beta_{i,t}}{\sigma_i} \mathbf{1}_{\{0\}}(z) + \frac{S_T^i \gamma_{i,t,z}}{z} \mathbf{1}_{\mathbb{R}_0}(z)$ *for* q_i*-a.e.* $(t,z) \in [0,T] \times \mathbb{R}$ *and*
$D_{t,z}^i S_T^j = 0, i \neq j$

An explicit representation of LRM for $(S^1 - S^2)^+$ is given as follows:

Proposition 4.3

For any $t \in [0, T]$, *we have*

$$\xi_t^{1,(S_T^1 - S_T^2)^+} = \frac{1}{S_{t-}^1 \left(\beta_{1,t}^2 + \int_{\mathbb{R}_0} \gamma_{1,t,z}^2 \nu_1(dz)\right)} \Bigg\{ \beta_{1,t}^2 \mathbb{E}_{\mathbb{P}^*}[\mathbf{1}_{\{S_T^1 > S_T^2\}} S_T^1 | \mathcal{F}_{t-}] + \int_{\mathbb{R}_0} \mathbb{E}_{\mathbb{P}^*}[(S_T^1(1+\gamma_{i,t,z}) - S_T^2)^+ - (S_T^2 - S_T^2)^+ | \mathcal{F}_{t-}] \gamma_{1,t,z} \nu_1(dz) \Bigg\}$$

and

$$\xi_t^{2,(S_T^1 - S_T^2)^+} = \frac{1}{S_{t-}^2 \left(\beta_{2,t}^2 + \int_{\mathbb{R}_0} \gamma_{2,t,z}^2 \nu_2(dz)\right)} \Bigg\{ -\beta_{2,t}^2 \mathbb{E}_{\mathbb{P}^*}[\mathbf{1}_{\{S_T^1 > S_T^2\}} S_T^2 | \mathcal{F}_{t-}] + \int_{\mathbb{R}_0} \mathbb{E}_{\mathbb{P}^*}[-(S_T^2(1+\gamma_{2,t,z}) + S_T^1)^+ - (S_T^1 - S_T^2)^+ | \mathcal{F}_{t-}] \gamma_{2,t,z} \nu_2(dz) \Bigg\}.$$

Future research

1. LRM for general multidimensional jump diffusion model:

$$\begin{cases} dS_t^i = S_{t-}^i \left[\alpha_t^i dt + \sum_{l=1}^d \beta_t^{i,l} dW_{l,t} + \sum_{l=1}^d \int_{\mathbb{R}_0} \Gamma_t^{i,l}(z) \tilde{N}_l(dt, dz) \right] \\ S_0^i \in \mathbb{R}_{++}, i = 1, \cdots, d \end{cases}$$

   where α, β and γ are predictable process.

2. Numerical analysis on LRM for multidimensional Lévy markets.

See also my website:
https://sites.google.com/site/ryoichisuzukifinance/

Thank you for your attention!