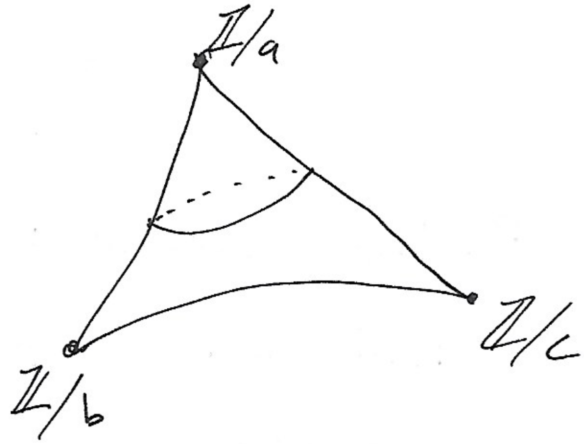
Quantum cohomology of orbifold spheres

BU-Keio Workshop, 06/29/2017

(Based on joint work with Cho-Hong-Lau)

- Today I'll give a description of the quantum cohomology of a sphere with three orbifolds points.

We denote $\mathbb{P}^1_{a,b,c}$:



$\mathbb{P}^1_{a,b,c}$ can be constructed as Σ/G, where Σ is a Riemann surface and G a finite group.

The main

<u>thm</u>: There exist a ring isomorphism

$$KS : QH^*(\mathbb{P}^1_{a,b,c}) \longrightarrow Jac(W_{a,b,c}),$$

where Jac stands for Jacobian ring and $W_{a,b,c}$ is a certain power series.

Let's explain these concepts:

(I) Quantum cohomology: $QH^*(\mathbb{P}^1_{a,b,c})$

As a vector space: inertia orbifold

$$QH^*(\mathbb{P}^1_{a,b,c}) = H^*(I\mathbb{P}^1_{a,b,c}, \Lambda)$$

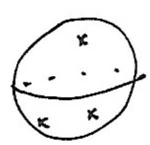
Novikov field.

$$= H^*(S^2, \Lambda) \oplus \Big(\bigoplus_{g \in \mathbb{Z}/a \setminus \{0\}} \Lambda\Big) \oplus \Big(\bigoplus_{g \in \mathbb{Z}/b \setminus \{0\}} \Lambda\Big)$$
$$\oplus \Big(\bigoplus_{g \in \mathbb{Z}/c \setminus \{0\}} \Lambda\Big).$$

Product structure:

Pick a.c.s. compatible with symplectic form and consider

$$\mathcal{M}_3 = \left\{ \begin{array}{l} \text{moduli space of holomorphic} \\ \text{spheres with 3-marked pts} \end{array} \right\}$$

$= \Big\{$  $\xrightarrow{u} \mathbb{P}^1_{a,b,c} \Big\}$

There are evaluation maps

$$I\mathbb{P}^1_{a,b,c} \xleftarrow{ev_1 \times ev_2} \mathcal{M}_3 \xrightarrow{ev_0} I\mathbb{P}^1_{a,b,c}$$

<u>Rmk</u>: We consider <u>representable</u> holomorphic maps which allows us to lift the evaluation maps from $\mathbb{P}^1_{a,b,c}$ to $I\mathbb{P}^1_{a,b,c}$:

A pull-push construction using the above diagram defines a map

$$* : QH^*(\mathbb{P}^1_{a,b,c})^{\otimes 2} \longrightarrow QH^*(\mathbb{P}^1_{a,b,c})$$

This map is associative by a theorem of Chen-Ruan.

The quantum cohomology is $H^*(I\mathbb{P}^1_{a,b,c})$ equipped with the product $*$.

<u>Rmk</u>: This product is also commutative and quantum cohomology is in fact a Frobenius algebra.

(II) Jacobian ring: $Jac(W_{a,b,c})$

Novikov field: $\Lambda = \left\{ \sum_{i=0}^{\infty} a_i T^{\lambda_i},\ a_i \in \mathbb{C},\ \lambda_i \in \mathbb{R},\ \lambda_i \nearrow \infty \right\}$.

$$\Lambda \langle\langle x, y, z \rangle\rangle = \left\{ \sum_{i,j,k \geq 0} c_{i,j,k}\, x^i y^j z^k \ \middle|\ \cdots \right.$$

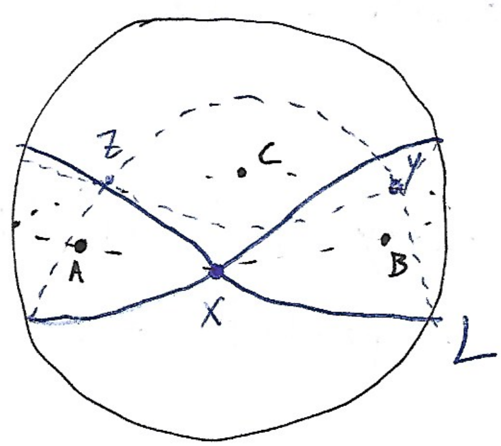
... $c_{i,j,k} \in \Lambda$ and $\mathrm{val}(c_{i,j,k}) \xrightarrow[i+j+k \to \infty]{} \infty$.

We define

$$\mathrm{Jac}(W) = \frac{\Lambda\langle\langle x, y, z\rangle\rangle}{\langle \partial_x W, \partial_y W, \partial_z W\rangle} \quad \text{for any } W \in \Lambda\langle\langle x,y,z\rangle\rangle.$$

<u>Definition of $W_{a,b,c}$</u>:

Consider the following immersed circle in $\mathbb{P}^1_{a,b,c}$:



A, B, C : orbifold pts.

L : . 3 transverse self-intersections

. Invariant under reflection on equator.

. "Monotone": divides sphere into 5 regions three bigons of area 2 and two triangles of area 1. (Total area = 8).

Following Fukaya-Oh-Ohta-Ono and Akaho-Joyce there is a curved A_∞-algebra on

$$C^*(L) = C^*(S^1) \oplus \Lambda^{\oplus 2}\langle X\rangle \oplus \Lambda^{\oplus 2}\langle Y\rangle \oplus \Lambda^{\oplus 2}\langle Z\rangle$$

denoted by $\mathcal{F}(L) = \left(C^*(L), \{m_k\}_{k\geq 0}\right)$.

Consider the Maurer-Cartan equation on $\mathcal{F}(L)$

$$\sum_{k\geq 0} m_k(b,\dots,b) = \underbrace{\mathcal{P}(b)}_{\text{element of } \Lambda} \cdot \underbrace{\mathbb{1}_L}_{\text{unit in } \mathcal{F}(L)}.$$

$b \in \mathcal{F}(L)^{odd}$.

<u>lemma</u>: Take $b = xX + y\cancel{Y} + zZ$ with $x,y,z \in \Lambda_+$. Any such b solves the Maurer-Cartan equation.

<u>Conclusion</u>: $\mathcal{P}(b) = W(x,y,z) =: W_{a,b,c}$ is an element of $\Lambda\langle\langle x,y,z\rangle\rangle$

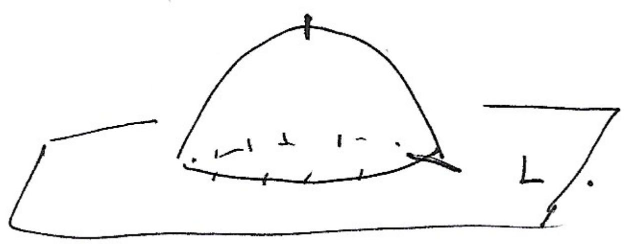
$W_{a,b,c}$ was computed by Cho-Hong-Kim-Lau.

Rmk: When $\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \geq 1$, $W_{a,b,c}$ is in fact a polynomial in x, y, z.

When $\frac{1}{a}+\frac{1}{b}+\frac{1}{c} < 1$, $W_{a,b,c}$ is a genuine power series.

III The map KS

Consider moduli space:



holomorphic disks with boundary on L and $(k+1)$-boundary marked points and one interior pt.

As before defines a map:

$$q_k : QH^*(\mathbb{P}^1_{a,b,c}) \otimes \mathcal{F}(L)^{\otimes k} \longrightarrow \mathcal{F}(L)$$

Define $q : QH^*(\mathbb{P}^1_{a,b,c}) \longrightarrow \mathcal{F}(L)$

by $q(\gamma) = \sum_{k \geq 0} q_k(\gamma, b, \ldots, b)$.

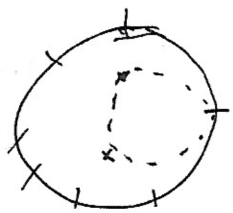
<u>lemma</u>: $q(\gamma) = KS(\gamma) \cdot 1_L$

$KS(\gamma)$ ↓ element in Λ

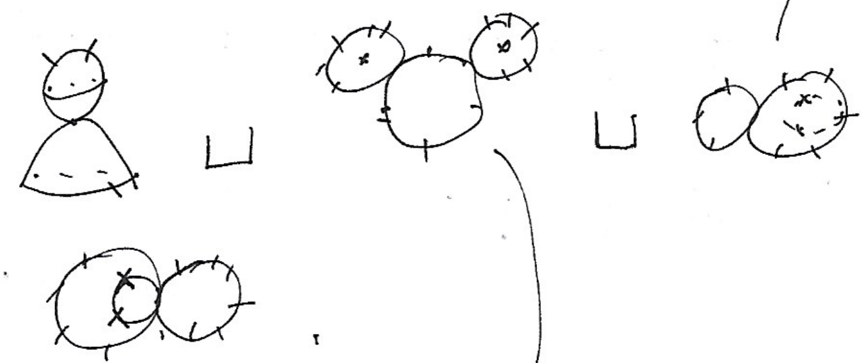
We define $KS(\gamma) \in \Lambda\langle\langle x, y, z \rangle\rangle$ using this lemma.

<u>thm</u>: KS is a ring map.

Idea: Consider moduli space



boundary: $\sqcup$ $\sqcup$ $\sqcup$



$KS \circ \left(\text{product in } QH\right)$

$\text{product in Jac} \circ (KS \otimes KS)$

Contained in $\mathrm{Im}(m_1^b)$

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follows from the fact that X, Y, Z generate $F(L)$ as an algebra. //

lemma: $KS([A]) = x$ + higher energy

$KS([B]) = y$ + "

$KS([C]) = z$ + " .

Upshot: lemma + KS ring map $\Longrightarrow$ KS surjective.

Finally we need: $\mathrm{rk}\ \mathrm{Jac}(W_{a,b,c}) = \mathrm{rk}(QH^*(\mathbb{P}^1_{a,b,c}))$

$= a+b+c-1$.

When i) $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} > 1$: $W_{a,b,c}$ is Morse with $a+b+c-1$ critical pts.

ii) $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1$: Compute $\mathrm{Jac}(W_{a,b,c})$ directly

iii) $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} < 1$: Deform $W_{a,b,c}$ to $W_0 := x^a + y^b + z^c - T^{-8} xyz$ in a flat family and compute.

$\therefore$ $KS : QH^*(\mathbb{P}^1_{a,b,c}) \xrightarrow{\sim} \mathrm{Jac}(W_{a,b,c})$ ring isomorphism.