Entropy of an autoequivalence on Calabi–Yau manifolds
(arXiv:1704.06957)

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Plan

- Topological entropy and Gromov–Yomdin theorem.
- Background on autoequivalences.
- Categorical entropy and Kikuta–Takahashi conjecture.
- Reason to expect counterexamples via Homological Mirror Symmetry.
- Counterexamples.
Topological entropy

Definition

$\topo(h(f))$ measures "how fast points spread out when iterate $f$".

$N(n,\epsilon) := \max \{ \# F : F \subset X, \max_{0 \leq i \leq n} d(f^i(x), f^i(y)) \geq \epsilon \text{ for any } x, y \in F \}$. 

$\topo(h(f)) := \lim_{\epsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{\log N(n,\epsilon)}{n} \in [0, \infty]$. 
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Properties

$h_{\text{top}}(f)$ is a topological invariant: If $(X, d) \overset{\sim}{=} (X, d')$, then one gets the same topological entropy.

$f^n = \text{id}_X \Rightarrow h_{\text{top}}(f) = 0$.

Theorem (Gromov, Yomdin)

$X$ compact Kähler manifold, $f: X \to X$ holomorphic surjective.

$h_{\text{top}}(f) = \log \rho(f^*)$.

Here $\rho$ is the spectral radius of $f^*: H^* (X; \mathbb{C}) \to H^* (X; \mathbb{C})$. 
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- Bondal–Orlov ’01: When \( K_X \) is (anti-)ample, the group of autoequivalences is generated by the standard ones.
Examples of autoequivalences

Spherical twists (Seidel–Thomas '01)

Let $X$ be a Calabi–Yau manifold of dimension $d$.

$E \in D^{b}(X)$ is spherical if $\text{Hom}(E, E[\ast]) \simeq H^{\ast}(S^{d}; \mathbb{C})$.

e.g. Lagrangian sphere in derived Fukaya category.
e.g. $O_X$ is spherical iff $H^{i}(O_X) = 0$ for $0 < i < d$, i.e. $X$ is strict CY.
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Autoequivalences from monodromies

- Kähler moduli of CY hypersurface $X \subset \mathbb{P}^{d+1}$:

  ![Diagram showing Kähler moduli with points labeled Large Volume Limit, Conifold point, and Gepner point.]

- Monodromies $\leadsto$ Autoequivalences on $\mathcal{D}^b(X)$

- Kontsevich ’96, Horja ’99:
  LVL $\leadsto \otimes \mathcal{O}(1)$, Conifold $\leadsto T_{\mathcal{O}_X}$, Gepner $\leadsto T_{\mathcal{O}_X} \circ \otimes \mathcal{O}(1)$.

- Ballard–Favero–Katzarkov ’12: $(T_{\mathcal{O}_X} \circ \otimes \mathcal{O}(1))^{d+2} = [2]$. 
Categorical entropy

Results

Entropy: Measures "complexity" of an autoequivalence.

\[ e.g. \otimes O(1), T^{O_X}, T^{O_X} \circ \otimes O(1) \text{ all have zero entropy.} \]

Theorem (\( d \geq 3 \)) \( T^{O_X} \circ \otimes O(-1) \) has positive entropy.

Its exponential is the unique \( \lambda > 1 \) satisfying

\[ \sum_{k \geq 1} \chi(O(k)) \lambda^k = 1. \]

(e.g. quintic CY3: \( \lambda^4 - 9 \lambda^3 + 11 \lambda^2 - 9 \lambda + 1 = 0. \))
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\[ \text{Theorem (} d \geq 3) \quad T_{\mathcal{O}_X} \circ \otimes \mathcal{O}(1)^{-1} \text{ has positive entropy.} \]

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$T_{O_X} \circ \otimes O(-1)$ has positive entropy. Its exponential is the unique $\lambda > 1$ satisfying

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\( \implies \) Counterexamples of Kikuta–Takahashi conjecture.
Categorical entropy
Definition (Dimitrov–Haiden–Katzarkov–Kontsevich '13)
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**Definition**
For $E, F \in \mathcal{D}$, the *complexity* of $F$ relative to $E$ is

\[
\delta(E, F) := \inf \left\{ k \mid \begin{array}{ccc}
0 & \rightarrow & A_1 \\
\downarrow & & \downarrow \\
E[n_1] & \rightarrow & F \oplus F'
\end{array} \cdots \begin{array}{ccc}
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**Definition**
If $\mathcal{D}$ has a split generator $G$, then the *categorical entropy* of an autoequivalence $\Phi$ is

$$h_{\text{cat}}(\Phi) := \lim_{n \to \infty} \frac{\log \delta(G, \Phi^n G)}{n} \in [-\infty, \infty).$$
Categorical entropy

Properties

The limit exists. And is independent of the choice of $G$.

$\Phi^n = [m] = \Rightarrow h_{\text{cat}}(\Phi) = 0.$

Conjecture (Kikuta–Takahashi)

For $D = D_b(X)$ and $\Phi$ an autoequivalence on $D$, $h_{\text{cat}}(\Phi) = \log \rho(\Phi_{H^*})$.

Proved: $\dim X = 1$; standard autoequivalences.
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Reason to expect counterexamples

Thurston: examples of pseudo-Anosov maps on Riemann surface $S$ ($g > 1$) that act trivially on $H^\ast$. These maps are symplectomorphisms, but not holomorphic.

Gromov–Yomdin fails in these cases: $\text{htop}(f) = \log \lambda > 0 = \log \rho(f^\ast)$.

DHKK: $\text{hcat}(f^\ast) = \log \lambda > 0$. Here $f^\ast$ is the induced autoequivalence on $\text{Fuk}(S)$.

Idea: If there are autoequivalences on $\text{Fuk}(X)$ with $\text{hcat}(\Phi) > \log \rho(\text{HH}^\bullet(\Phi))$ for some Calabi–Yau $X$, then by homological mirror symmetry, one may expect to find counterexamples of the conjecture on the mirror.
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Counterexamples

Theorem \( d \geq 3 \)
\[
\Phi := \text{OTO} \circ \otimes \text{O}(-1)
\]
has positive categorical entropy. Its exponential is the unique \( \lambda > 1 \) satisfying
\[
\sum_{k \geq 1} \chi(O(k)) \lambda^k = 1.
\]

Claim \( d \geq 4 \) even.
\( X \subset \mathbb{CP}^{d+1} \)
CY hypersurface of degree \( d+2 \).
Then \( \rho(\Phi H^*) = 1 \).

Hence \( h_{\text{cat}}(\Phi) > 0 = \log \rho(\Phi H^*) \).

So Kikuta–Takahashi conjecture fails in this case.
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**Claim**

\(d \geq 4 \text{ even. } X \subset \mathbb{CP}^{d+1} \text{ CY hypersurface of degree } d + 2. \text{ Then} \]

\[ \rho(\Phi_{H^*}) = 1. \]
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$\implies (T_{\mathcal{O}_X} \circ \otimes \mathcal{O}(1))^{d+2}_{H^*} = \text{id}_{H^*}$. 
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- Fact: $(T_2^2)_{H^*} = \text{id}_{H^*}$ when $X$ is of even dimension.
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- Fact: $(T_{S}^{2})_{\ast} = \text{id}_{\ast}$ when $X$ is of even dimension.
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  \[ \implies \rho(\Phi_{H^*}) = 1. \]
Sketch of proof of Theorem

DHKK: If $G$ and $G'$ are both split generators of $D^b(X)$, then

$h_{\text{cat}}(\Phi) = \lim_{n \to \infty} \frac{1}{n} \log \sum_{a \in \mathbb{Z}} \text{dim} \text{Hom}(G, \Phi_n G'[a])$.

Orlov: $G = \bigoplus_{d+1}^{\infty} O(i)$ and $G' = \bigoplus_{d+1}^{\infty} O(-i)$ are split generators.

Lemma: Recursive formula for the dimension of $\text{Hom}(O, \Phi_n (G' \otimes O(-k)[a]))$ via Kodaira vanishing.

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- $+$ some combinatorics $\Rightarrow$ Theorem.
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Thank you!