Entropy of an autoequivalence on Calabi–Yau manifolds

(arXiv:1704.06957)

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June 2017

BU-Keio workshop

Plan

- Topological entropy and Gromov–Yomdin theorem.
- Background on autoequivalences.
- Categorical entropy and Kikuta–Takahashi conjecture.
- Reason to expect counterexamples via Homological Mirror Symmetry.
- Counterexamples.

Topological entropy

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- Topological entropy $h_{\mathrm{top}}(f)$ measures "how fast points spread out when iterate f".
- $N(n, \epsilon) := \max\{\#F : F \subset X, \max_{0 \leq i \leq n} d(f^i(x), f^i(y)) \geq \epsilon$ for any $x, y \in F\}$.

Definition

$$h_{\mathrm{top}}(f) := \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{\log N(n, \epsilon)}{n} \in [0, \infty].$$

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Theorem (Gromov, Yomdin)

X compact Kähler manifold, $f : X \to X$ holomorphic surjective.

$$h_{\mathrm{top}}(f) = \log \rho(f^*).$$

Here ρ is the spectral radius of $f^* : H^*(X; \mathbb{C}) \to H^*(X; \mathbb{C})$.

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- Standard autoequivalences: $\otimes\mathcal{L}$, $\mathrm{Aut}(X)$, $[n]$.
- Bondal–Orlov '01: When K_X is (anti-)ample, the group of autoequivalences is generated by the standard ones.

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e.g. $\mathcal{O}_X$ is spherical iff $H^i(\mathcal{O}_X) = 0$ for $0 < i < d$, i.e. X is *strict* CY.

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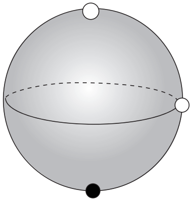
  e.g. Dehn twist along Lagrangian sphere.

Autoequivalences from monodromies

- Kähler moduli of CY hypersurface $X \subset \mathbb{CP}^{d+1}$:



- Monodromies $\rightsquigarrow$ Autoequivalences on $\mathcal{D}^b(X)$
- Kontsevich '96, Horja '99:
  LVL $\rightsquigarrow \otimes\mathcal{O}(1)$, Conifold $\rightsquigarrow \mathrm{T}_{\mathcal{O}_X}$, Gepner $\rightsquigarrow \mathrm{T}_{\mathcal{O}_X} \circ \otimes\mathcal{O}(1)$.
- Ballard–Favero–Katzarkov '12: $(\mathrm{T}_{\mathcal{O}_X} \circ \otimes\mathcal{O}(1))^{d+2} = [2]$.

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(e.g. quintic CY3: $\lambda^4 - 9\lambda^3 + 11\lambda^2 - 9\lambda + 1 = 0$.)

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$\Longrightarrow$ Counterexamples of Kikuta–Takahashi conjecture.

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For $E, F \in \mathcal{D}$, the *complexity* of F relative to E is

$$\delta(E,F) := \inf \left\{ k \;\middle|\; \begin{array}{ccccccc} 0 & \longrightarrow & A_1 & \cdots & A_{k-1} & \longrightarrow & F \oplus F' \\ & \nwarrow \quad \swarrow & & & & \nwarrow \quad \swarrow & \\ & E[n_1] & & \cdots & & E[n_k] & \end{array} \right\}$$

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Definition

If $\mathcal{D}$ has a split generator G, then the <u>categorical entropy</u> of an autoequivalence Φ is

$$h_{\mathrm{cat}}(\Phi) := \lim_{n\to\infty} \frac{\log \delta(G, \Phi^n G)}{n} \in [-\infty, \infty).$$

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For $\mathcal{D} = \mathcal{D}^b(X)$ and Φ an autoequivalence on $\mathcal{D}$,

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Proved: $\dim X = 1$; standard autoequivalences.

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- DHKK: $h_{\mathrm{cat}}(f_*) = \log \lambda > 0$. Here f_* is the induced autoequivalence on $\mathrm{Fuk}(S)$.
- Idea: If there are autoequivalences on $\mathrm{Fuk}(X)$ with $h_{\mathrm{cat}}(\Phi) > \log \rho(\mathrm{HH}_\bullet(\Phi))$ for some Calabi–Yau X, then by homological mirror symmetry, one may expect to find counterexamples of the conjecture on the mirror.

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Theorem ($d \geq 3$)

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Hence $h_{\mathrm{cat}}(\Phi) > 0 = \log \rho(\Phi_{H^*})$. So Kikuta–Takahashi conjecture fails in this case.

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- $+$ some combinatorics $\implies$ Theorem.

Thank you!