Boston-Keio Workshop

"The nonuniqueness of tangent cones at ∞ of Ricci-flat manifolds"

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§ Tangent cone at ∞

Def. $(X, d, p), (Y, d', q)$: pointed metric spaces

$r, \varepsilon > 0$.

f : (r, ε)-isometry from (X, d, p) to (Y, d', q)

$:\Longleftrightarrow$
(i) $f : B(p, r) \to Y$, $f(p) = q$
(ii) $|d(x, x') - d'(f(x), f(x'))| < \varepsilon \quad (\forall x, x' \in B(p, r))$
(iii) $B(q, r-\varepsilon) \subset B(\mathrm{Im} f, \varepsilon)$

Def $(X_n, d_n, p_n) \xrightarrow{n\to\infty} (Y, d', q)$ in p-GH topology
(pointed Gromov-Hausdorff)

$:\Longleftrightarrow \forall r, \varepsilon > 0 \ \exists N(r, \varepsilon)$: positive integer
s.t. $n \geq N(r, \varepsilon) \Rightarrow \exists f : (r, \varepsilon)$-isom. from (X_n, d_n, p_n) to (Y, d', q)

Def Let (X, d) be a (unbounded) metric space.

(Y, d', ξ) : tangent cone at ∞ of (X, d)

$:\Longleftrightarrow$ ${}^\exists p \in X,\ {}^\exists \varepsilon_n \downarrow 0$ s.t.

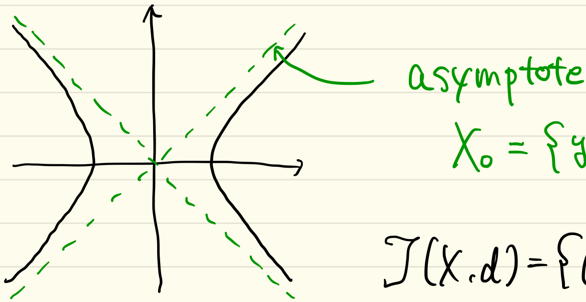
$$(X, \varepsilon_n d, p) \longrightarrow (Y, d', \xi)$$

$$\mathcal{T}(X, d) := \frac{\{(Y, d', \xi) : \text{tangent cone at } \infty \text{ of } (X, d)\}}{\text{pointed isometry}}$$

ex.

$X := \{(x,y) \in \mathbb{R}^2 \mid x^2 - y^2 = 1\}$

$d(x, x') := \|x - x'\|$



$X_0 = \{y = x\} \cup \{y = -x\}$

$\mathcal{T}(X, d) = \{(X_0, d, 0)\}$

ex. (ALE space)

$\Gamma \subset SU(2)$: finite subgroup.

$X \longrightarrow \mathbb{C}^2/\Gamma$: minimal resolution

Thm (Kronheimer)

$\exists g$: complete hyperkähler metric on X

s.t. $\mathcal{T}(X, d) = \{(\mathbb{C}^2/\Gamma, \text{Euclidean met.}, 0)\}$.

§ Existence and Uniqueness

(X^n, g) : complete Riem. mfd of $\dim_{\mathbb{R}} = n$.

$d = d_g$: Riem. distance.

Assumption		Conclusion
⓪ $\mathrm{Ric}_g \geq 0$	$\Longrightarrow$	$\mathcal{T}(X,d) \neq \phi$. (∵ Gromov's cptness thm)
① g : maximal volume growth ($\mathrm{Vol}(B(p,r)) \sim r^n$)	$\underset{⓪①}{\Longrightarrow}$ (Cheeger-Colding)	$\forall (Y, d', q) \in \mathcal{T}(X,d)$ is a metric cone of $\exists (S, d_S)$ ((S, d_S) : cross section of Y)
② $\mathrm{Ric}_g \equiv 0$ ③ $\exists (Y, d', q) \in \mathcal{T}(X,d)$ s.t. cross sec. is smooth	$\underset{①②③}{\Longrightarrow}$	$\# \mathcal{T}(X,d) = 1$. (Colding-Minicozzi)

§ Nonuniqueness

Thm (Perelman, Colding-Naber)

$\exists\ (X, g)$ with ⓪①③

s.t. $\quad \# \mathcal{T}(X, d) > 1.$

(contains 1-para. family)

Thm (H)

$\exists\ (X^4, g)$ with ②③

s.t. $\quad \# \mathcal{T}(X, d) > 1.$

Outline of proof

Thm (Gibbons-Hawking, Anderson-Kronheimer-LeBrun, Goto)

Let $\Lambda \subset \mathbb{R}^3$ be countable with

$$\sum_{\lambda \in \Lambda} \frac{1}{1 + \|\lambda\|} < +\infty .$$

Then $\exists\, (X^4, g)$: complete hyperkähler mfd

($\Rightarrow Ric_g \equiv 0$)

with a Riem. submersion

$$\mu : (X, g) \longrightarrow (\mathbb{R}^3, \Phi_\Lambda \cdot h_0)$$

where

$$\begin{cases} \Phi_\Lambda(x) := \sum_{\lambda \in \Lambda} \frac{1}{\|x - \lambda\|} \\ h_0 : \text{Euclidean metric on } \mathbb{R}^3 \end{cases}$$

Denote by d_S^T $(0 \leq S < T \leq \infty)$ the distance

on $\mathbb{R}^3$ induced by

$$\left(\int_S^T \frac{dt}{\|x - (t^2, 0, 0)\|} \right) \cdot h_0$$

Take $\Lambda \subset \mathbb{R}^3$ appropriately $\rightsquigarrow (X, g)$

$$\begin{tikzcd}
\mathcal{T}(X,d) \cong & \{(\mathbb{R}^3, d_S^\infty, 0)\}_{0<S<\infty} \arrow[r, "S \to 0"] \arrow[dl, "S \to \infty"'] & (\mathbb{R}^3, d_0^\infty, 0) \\
(\mathbb{R}^3, h_0, 0) & & \{(\mathbb{R}^3, d_0^T, 0)\}_{0<T<\infty} \arrow[u, "T \to \infty"'] \arrow[d, "T \to 0"] \\
\{(\mathbb{R}^3, h_\theta, 0)\}_{0<\theta<1} \arrow[u, "\theta \to 0"] \arrow[rr, "\theta \to 1"'] & & (\mathbb{R}^3, h_1, 0)
\end{tikzcd}$$

$\left(h_1 = \frac{1}{\|x\|} \cdot h_0 \right)$

$\left(h_\theta = (1-\theta) h_0 + \theta h_1 \right)$