Construction of Lefschetz fibrations and pencils via mapping class groups

Kenta Hayano (Keio University)

June 26, 2017 @ Boston University

Joint work w/ Refik İnanç Baykur (University of Massachusetts)

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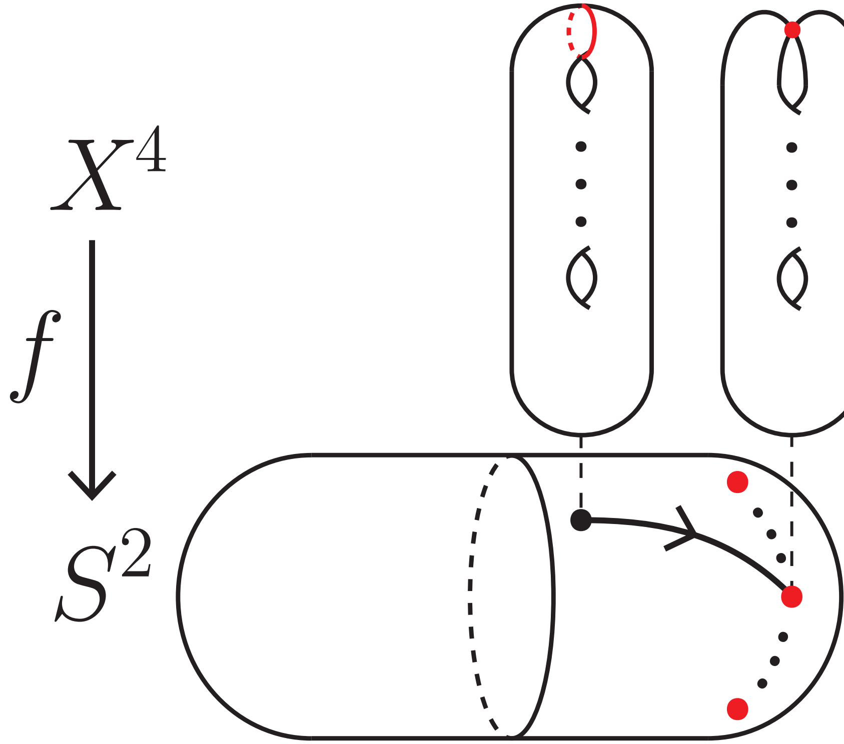
§.0 Outline of this talk

$*$ What is a Lefschetz fibraion?

$\rightarrow$ $\boldsymbol{f}$ w/ *good* critical pt's

(called a Lefschetz singularity)



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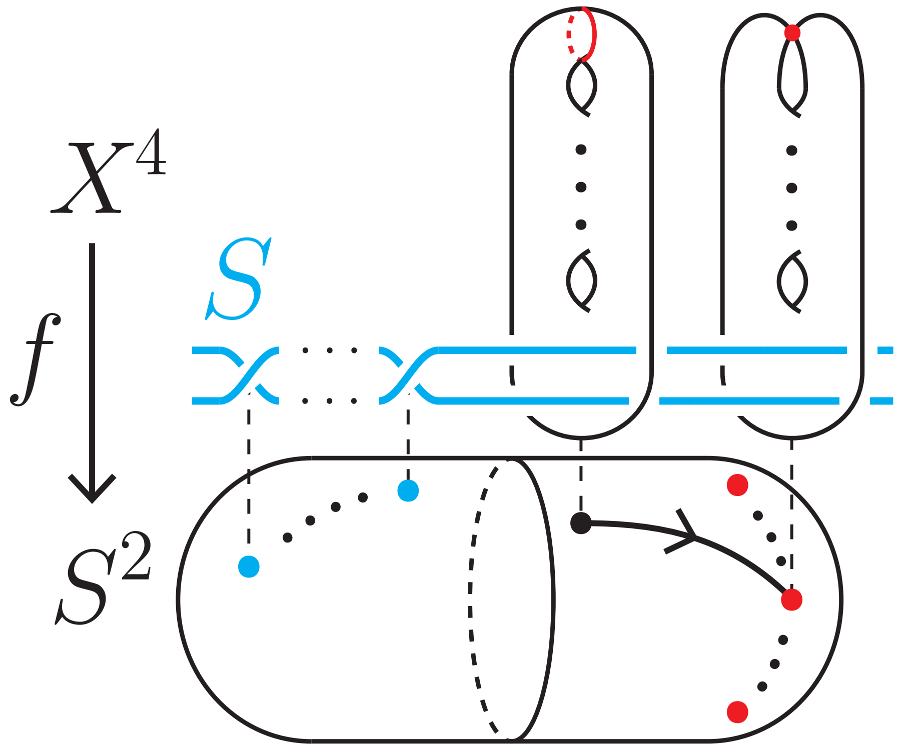
$\rightarrow$ $\boldsymbol{S}$: embedded surface s.t.

$\boldsymbol{f}|_{\boldsymbol{S}}$: simple branched cvr.

(+ some conditions...)



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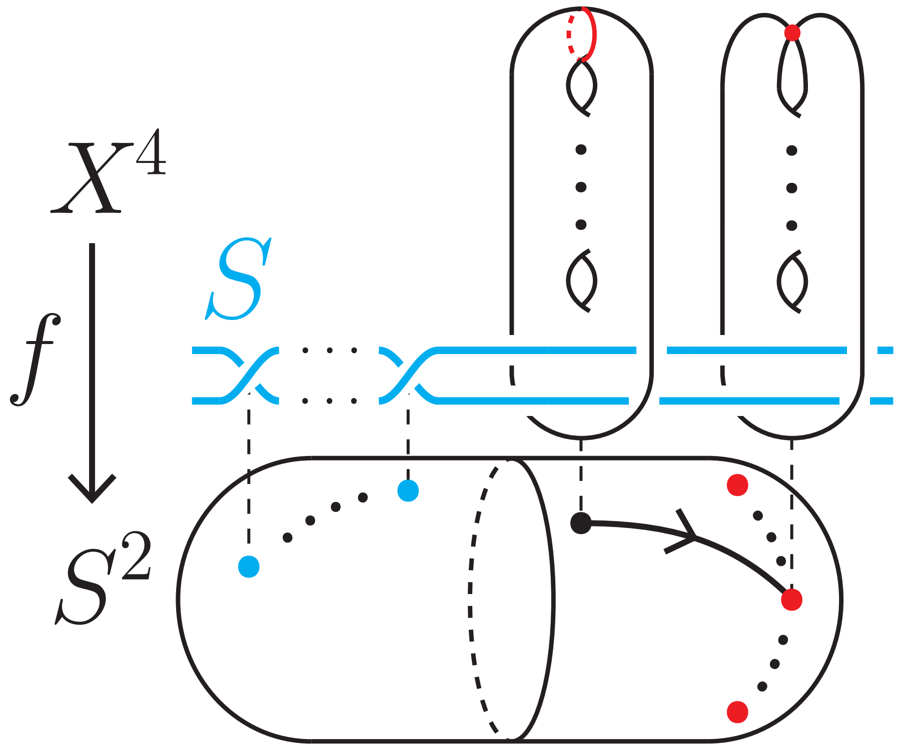

$*$ Roughly speaking, we obtained the following correspondence:

an LF w/ a multisection	$\longleftrightarrow$	an equality in a mapping class group of a surface w/ $\boldsymbol{\partial}$

$\diamondsuit$ Why should we care LFs?

$\Longrightarrow$Roughly, LFs are related to symplectic topology:

- $f : X \to S^2$: LF (w/ crit. pt) $\longrightarrow$ symplectic str. on X (Gompf)
- symp. str. on X $\longrightarrow$ LF on a blow-up of X (Donaldson)

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$*$ LFs are originally studied in complex/algebraic geometry:

- $L \to X$: very ample line bundle on a complex surface
  $\Longrightarrow$ generic pencil in $|L|$ is a *Lefschetz pencil*.
- Elliptic fibrations w/o multiple fibers are typical ex's of LFs.

$\diamondsuit$ Why should we care multisections?

1. Multisections are related to smooth invariants of 4-mfd's.

   (Taubes, Donaldson-Smith, Usher)

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2. In some sense, multisections reflect the topology of an LF.

   Indeed, using multisections we can:

   - construct counterex's to "the Stipsicz conjecture" on LFs.
   - construct an exotic pair of surfaces in a 4-manifold.
   - construct pairs of non-isomorphic LFs.
   - construct exotic pairs of LPs.

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$\diamondsuit$ Plan of this talk

§.1 Multisections and mapping class groups

§.2 Examples of LF with multisections

§.3 Applications

$*$ We will assume

- Manifolds : closed, smooth, oriented, connected.
- Maps between manifolds : smooth.

unless otherwise noted.

§.1 Multisections and mapping class groups

$f : X^4 \to S^2$, $\mathrm{Crit}(f) := \{x \in X \mid df_x$: **NOT** surj.$\}$

Definition $f : X^4 \to S^2$ is a **Lefschetz fibration** (**LF**) if:

(a) $^\forall q \in \mathrm{Crit}(f)$, $f(z, w) = z^2 + w^2$ under some complex coordinates around q & $f(q)$ compatible with orientations.

(b) $f|_{\mathrm{Crit}(f)}$: injective.

(c) No fibers contain spheres with self-intersection -1.

The genus of a regular fiber is the **genus** of f.

Definition $f\colon X \to S^2$: genus-g LF

$S \subset X$: embedded surface is a **multisection** or p-**section** if:

(a) $f|_S$: a p-fold simple branched cover.

(b) ${}^{\forall}q \notin \mathrm{Crit}(f)$: branched point of $f|_S$ is *positive*

(c) S is *compatible* with Lefschetz singularities

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i.e. $df_q : NS_q \to T_{f(q)}S^2$ preserves the orientations.

(NS : normal bundle of S)

($\Longleftrightarrow$ the monodromy around $f(q)$ is a **positive** half twist.)

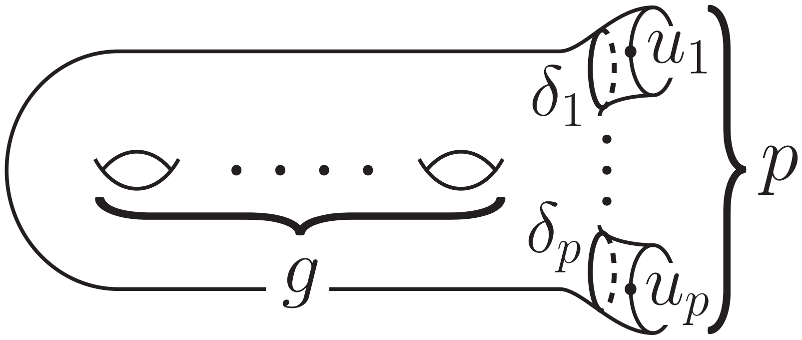
(c) S is *compatible* with Lefschetz singularities

i.e. ${}^\forall q \in S \cap \mathrm{Crit}(f)$, $\begin{matrix} {}^\exists(U,\varphi) \\ {}^\exists(V,\psi) \end{matrix}$: complex coordinates of $\begin{matrix} X \\ S^2 \end{matrix}$ at $\begin{matrix} q \\ f(q) \end{matrix}$ s.t.

$$\begin{array}{ccc} (U, U \cap S) & \xrightarrow{\varphi} & (\mathbb{C}^2, \Delta_{\mathbb{C}^2}) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle (z,w)\mapsto zw} \\ V & \xrightarrow{\psi} & \mathbb{C} \end{array}$$

where $\Delta_{\mathbb{C}^2} = \{(z,z) \in \mathbb{C}^2\}$

◇ Mapping class groups

$\Sigma_g^p =$ 

$U = \{u_1, \ldots, u_p\}$

$\delta_i \subset \Sigma_g^p$: simple closed curve along $\partial \Sigma_g^p$

$$\mathrm{Diff}^+(\Sigma_g^p; U) = \left\{ T : \Sigma_g^p \xrightarrow{\cong} \Sigma_g^p \ : \ \text{orientation preserving} \ \middle| \ T(U) = U \right\}$$

$$\mathrm{Mod}(\Sigma_g^p; U) = \pi_0(\mathrm{Diff}^+(\Sigma_g^p; U))$$

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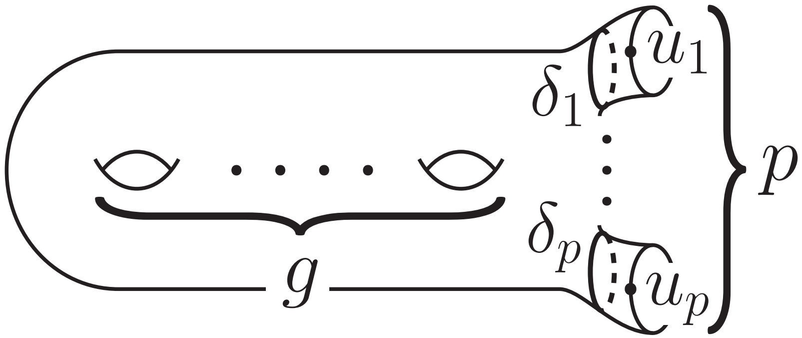
$\Sigma_g^p =$ 

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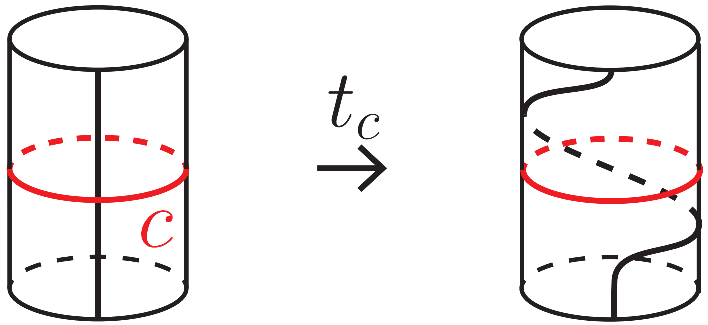
Remarks for experts of MCGs...

- $\varphi \in \mathrm{Diff}^+(\Sigma_g^p; U)$ may interchange ∂-comp's, in contrast with usual MCGs of surfaces w/ ∂.
- $t_{\delta_i} \in \mathrm{Mod}(\Sigma_g^p; U)$ is not trivial since an isotopy has to preserve u_i.

◇ **Important elements in** $\mathrm{Mod}(\boldsymbol{\Sigma_g^p}; \boldsymbol{U})$

1. $\boldsymbol{t_c} \in \mathrm{Mod}(\boldsymbol{\Sigma_g^p}; \boldsymbol{U})$

   for $\boldsymbol{c} \subset \mathbf{Int}(\boldsymbol{\Sigma_g^p})$: simple closed curve



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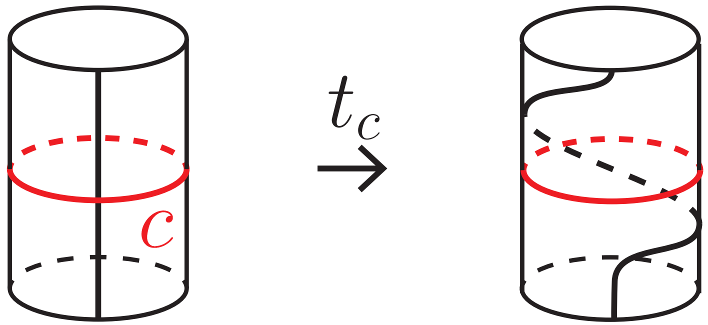
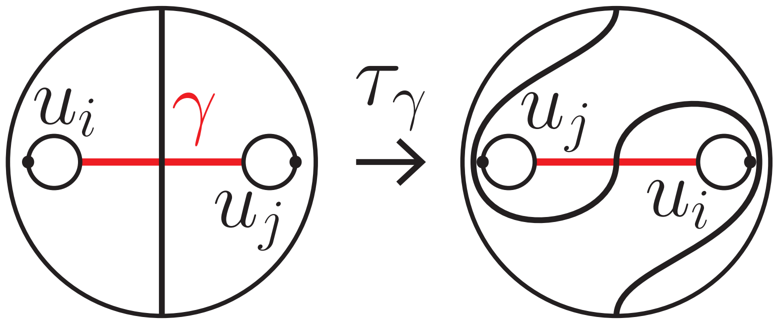


2. $\tau_\gamma \in \mathrm{Mod}(\Sigma_g^p; U)$

   for $\gamma \subset \Sigma_g^p$: path between ∂-comp's



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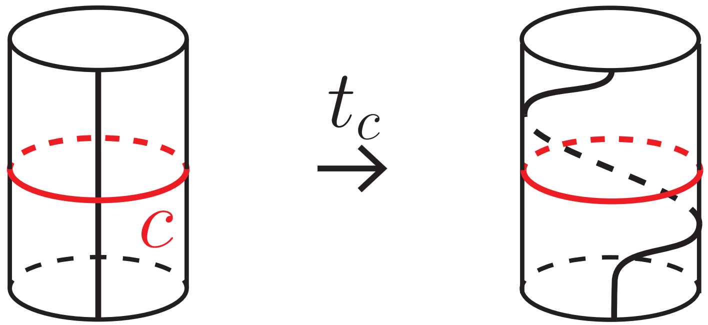

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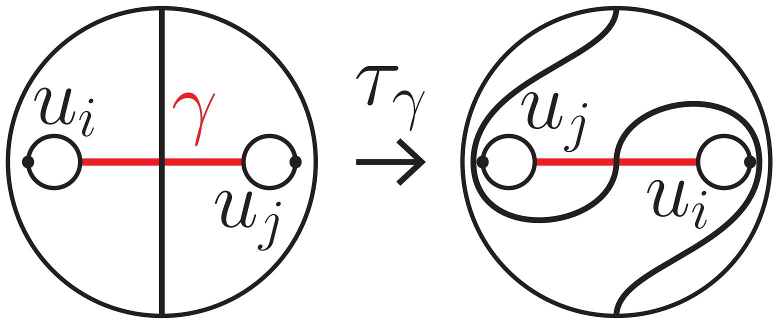
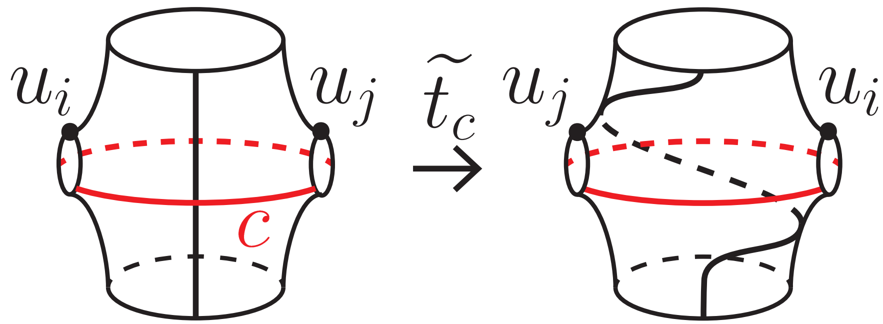


3. $\widetilde{t}_c \in \mathrm{Mod}(\Sigma_g^p; U)$

   for $c \subset \Sigma_g^p$: pair of paths between ∂-comp's



Theorem (Baykur-H.)

From an equality

$$\tau_{\gamma_1} \cdots \tau_{\gamma_k} \cdot \widetilde{t_{c_1}} \cdots \widetilde{t_{c_r}} \cdot t_{c_{r+1}} \cdots t_{c_l} = t_{\delta_1}^{a_1} \cdots t_{\delta_p}^{a_p} \qquad (1)$$

in $\mathrm{Mod}(\Sigma_g^p; U)$, we can construct

- $f : X \to S^2$: genus-g LF,
- S : p-sec. w/
  1. k branched points away from $\mathrm{Crit}(f)$, r branched points in $\mathrm{Crit}(f)$,
  2. self-intersection $-(\Sigma_{i=1}^p a_i) + 2k + r$.

∗ Conversely, a *monodromy* of f & S yields the equality (1).

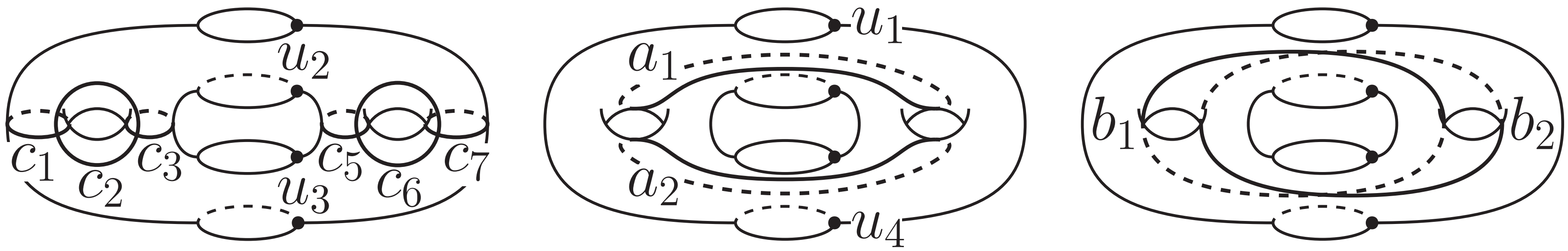
∗ Generalization of Kas ('80) & Matsumoto ('96)'s result.

(for Lefschetz fibrations without multisections)

§.2 Examples of LF with multisections

Example (Baykur-H.) The following holds in $\mathbf{Mod}(\boldsymbol{\Sigma_3^4}; \boldsymbol{U})$:

$$(t_1 t_3 t_5 t_7 t_2 t_6 t_{a_1} t_{a_2} t_{b_1} t_{b_2} t_1 t_3 t_5 t_7 t_{b1} t_{b_2} t_2 t_6)^2 = t_{\delta_1} t_{\delta_2} t_{\delta_3} t_{\delta_4}.$$



* $^{\exists}f_{1,1,1,1} : X \to S^2$: genus-$\mathbf{3}$ LF with four (disjoint) sections.

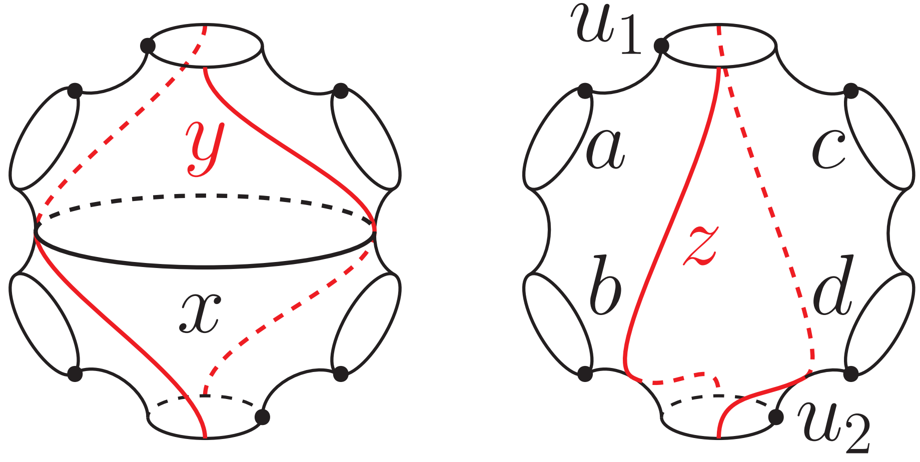
  (these are NOT multisections so far...)

* X is homeomorphic to $\mathbf{K3}\sharp 4\overline{\mathbb{CP}^2}$.

* To modify $f_{1,1,1,1}$ to LFs w/ multisections, we need:

Lemma (Baykur-H.) The following holds in $\mathbf{Mod}(\boldsymbol{\Sigma_0^6};\boldsymbol{U})$:

$$t_{\delta_2}^{-1}\widetilde{t_z}t_x\widetilde{t_y} = t_a t_b t_c t_d.$$

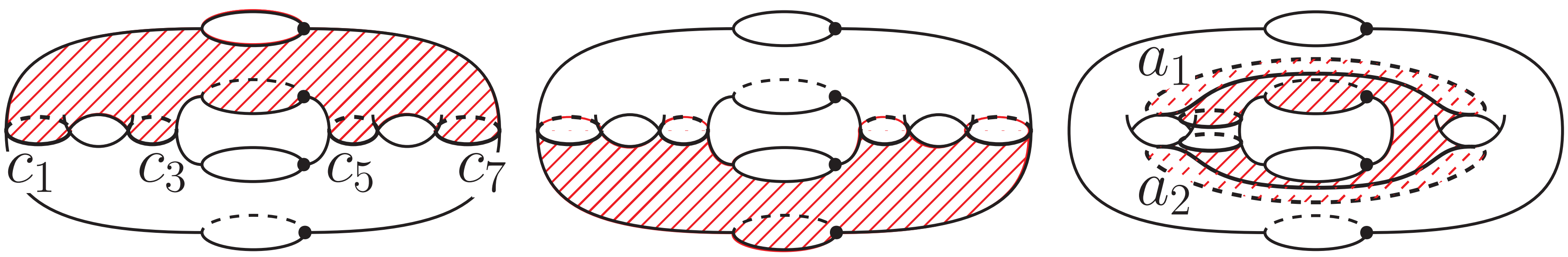


* Generalization of *Lantern relation* in $\mathbf{Mod}(\boldsymbol{\Sigma_0^4})$.

◇ Construction of multisections

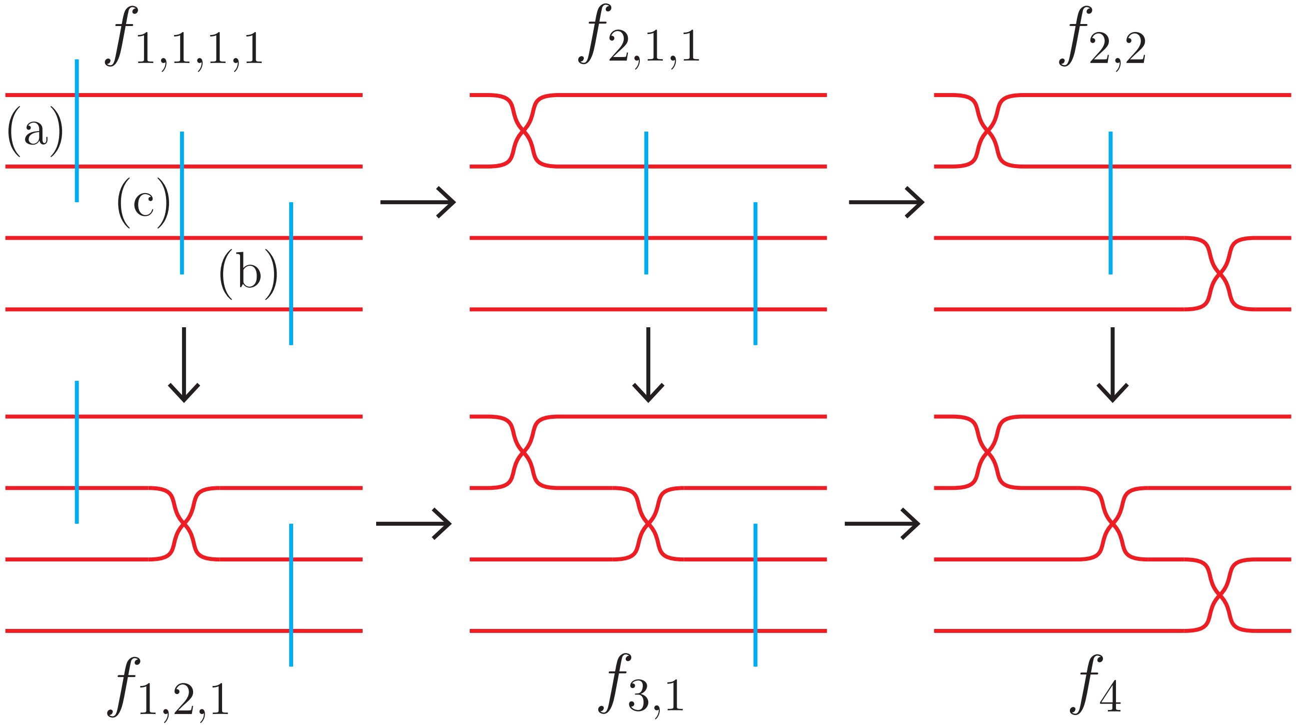
Apply the Lantern substitution at a, b and c, along the spheres (a), (b) and (c), respectively.

$$\begin{aligned} t_{\delta_4}t_{\delta_3}t_{\delta_2}t_{\delta_1} &= (t_1t_3t_5t_7t_2t_6t_{a_1}t_{a_2}t_{b_1}t_{b_2}t_1t_3t_5t_7t_{b1}t_{b_2}t_2t_6)^2 \\ &\sim \underbrace{t_1t_3t_5t_7}_{a}\, t_2t_6t_{a_1}t_{a_2}t_{b_1}t_{b_2}\, \underbrace{t_1t_3t_5t_7}_{b}\, t_{b_1}t_{b_2}t_2t_6 \\ &\quad \cdot t_1t_5t_7t_{t_3(c_2)}t_6\, \underbrace{t_3t_{a_1}t_{a_2}t_3}_{c}\, t_{b_1}t_{t_3^{-1}(b_2)}t_1t_5t_7t_{b_1}t_{b_2}t_2t_6. \end{aligned}$$



(a) (b) (c)

* We can obtain LFs with multisections using the Lemma.



Red : multisections, Blue : spheres (a), (b) and (c).

* The total space X' of $f_{2,2}$ is diffeomorphic to that of $f_{3,1}$. (cf. Gompf '95 and Endo '10)
* All the multisections are spheres w/ self-intersection -1.

§.3 Applications

◇ Counterexamples to the Stipsicz's conjecture

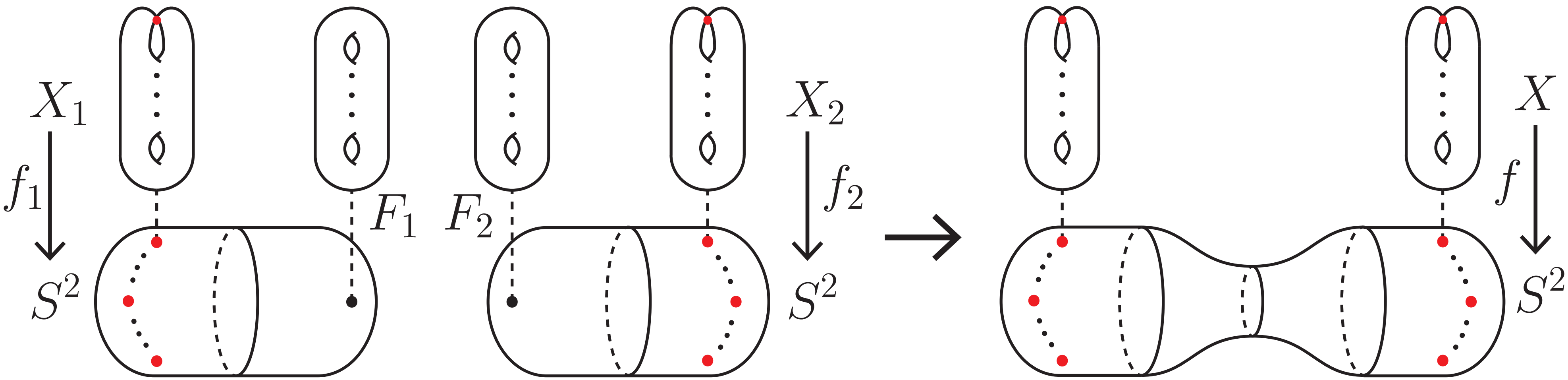
Definition

$f_i : X_i \to S^2$: genus-g LF $(i = 1, 2)$, $F_i \subset X_i$: reg. fiber

$\Longrightarrow X = (X_1 \setminus \nu F_1) \cup_\varphi (X_2 \setminus \nu F_2)$ admits a genus-g LF

(νF_i : tubular nbh. of F_i, $\varphi : \partial\nu F_1 \to \partial\nu F_2$: diffeo.)

$f = f_1 \sharp_f f_2$: new LF is called a **fiber-sum** of f_1 and f_2



Definition

f : LF is **fiber-sum indecomposable**

$\underset{\text{def}}{\iff} \nexists f_i$: LF w/ critical pts. $(i = 1, 2)$ s.t. $f = f_1 \sharp_f f_2$

Theorem (Stipsicz '00, Smith '01)

f : LF has a section w/ self-intersection -1

$\Longrightarrow f$ is fiber-sum indecomposable.

Conjecture (Stipsicz's Conjecture)

The converse is also true.

Theorem (Sato '08)

Stipsicz's conj. is false.

Precisely, $^{\exists}$genus-2 LF s.t.
- fiber-sum indecomposable.
- $^{\nexists}$section w/ self-int. -1

Theorem (Baykur-H.)

$f_{2,2}$ & f_4 are genus-3 counterex's to Stipsicz's conj.

$*$ Our examples are the first genus-3 counterexamples.

(genus-2 LF in Sato's thm. was the only counterexample.)

$*$ We can further construct counterex's to Stipsicz's conj. **w/ arbitarary genus-$g \geq 3$** in another way.

(by examining spin structures on LPs. Baykur-Monden-H. in preparation)

Theorem (Sato '08)

$f : X \to S^2$: genus-g LF, $F \subset X$: regular fiber

Suppose X is NOT rational or ruled (e.g. $b^+(X) \geq 3$).

$$\mathcal{E}_X = \left\{ [S] \in H_2(X;\mathbb{Z}) \;\middle|\; \begin{array}{l} S \subset X \text{ : sphere w/ self-int. } -1 \\ \omega|_S \geq 0 \end{array} \right\}$$

Then, $|\mathcal{E}_X| < \infty$, $F \cdot e \geq 1$ for ${}^\forall e \in \mathcal{E}_X$ and

$$\sum_{e \in \mathcal{E}_X} F \cdot e \leq 2g - 2.$$

$*$ For $f_{2,2} : X_{2,2} \to S^2$, $\mathcal{E}_{X_{2,2}} = \{E_1, E_2\}$ and $F \cdot E_1 = F \cdot E_2 = 2$.

$*$ For $f_4 : X_4 \to S^2$, $\mathcal{E}_{X_4} = \{E\}$ and $F \cdot E = 4$.

◇ An exotic pair of surfaces in a 4-manifold

Theorem (Baykur-H.) $F_{i,j} \subset X'$: a regular fiber of $f_{i,j}$.

- $(X', F_{2,2})$ and $(X', F_{3,1})$ are pairwise homeo, but not diffeo.
- $\exists \omega_{i,j}$: symp. form s.t. $\omega_{i,j}$ makes $F_{i,j}$ symplectic, $\omega_{2,2}$, $\omega_{3,1}$: deformation equivalent.

∗ Many exotic pairs are known.

(Finashin, Fintushel-Stern, Kim-Ruberman, Mark, etc.).

However,
– some of them are known to be non-symplectic,
– none of them are proved to be symplectic.

∗ The second statement immediately follows from

Mcduff-Symington's result on Gompf's symplectic sums.

* **Existence of homeo.**

1. Prove that $\pi_1(X' \setminus F_{i,j}) = 1$, $[F_{i,j}]$: NOT characteristic.
2. ${}^{\exists}\phi$: automorphism of $H_2(X')$ s.t. $\phi(F_{2,2}) = F_{3,1}$ (Wall).
3. ϕ can be realized by a self-homeo. φ of X' (Freedman).
4. $X' \setminus F_{3,1}, X' \setminus \varphi(F_{2,2})$: simply connected, $[F_{3,1}] = [\varphi(F_{2,2})]$ $\Longrightarrow \varphi(F_{2,2})$ and $F_{3,1}$ are topologically isotopic (Sunukujian).

* **Non-existence of diffeo.**

1. X' : NOT rational or ruled. Thus ${}^{\forall}\Phi : X' \xrightarrow[C^\infty]{\cong} X'$ preserves the homology classes E_1, E_2 of two exceptional spheres (Li).
2. $\{F_{3,1}\cdot E_1, F_{3,1}\cdot E_2\} = \{3, 1\}$, while $\{F_{2,2}\cdot E_1, F_{2,2}\cdot E_2\} = \{2, 2\}$.

Thank you for your attention!!

Summary

- Motivations to study multisections.
- The definition of Lefschetz fibrations.
- The definition of multisections.
- Relation between multisections and MCG.
- The braiding Lantern relation.
- Construction of Lefschetz fibrations with multisections.
- Counterexamples to the Stipsicz's conjecture.
- An exotic pair of surfaces in a 4-manifold.