Framed curves in the Euclidean space

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Abstract

As preparation for my presentation, we review the theory of framed curves in the Euclidean space. A framed curve is a space curve with a moving frame which may have singular points. The curvature of a framed curve is quite useful to analyse the framed curve and its singularities. In fact, we obtained the existence and the uniqueness for framed curves. As applications, we consider basic properties and some developable surfaces which come from framed curves.

1 Introduction

We define the **orthogonal** $(n-1)$**-frame** in $\mathbb{R}^n$

$$V_{n,n-1}=\{\boldsymbol{\nu}\in S^{n-1}\times\ldots\times S^{n-1} \mid \boldsymbol{\nu}_i\cdot\boldsymbol{\nu}_j=0\},$$

where $\boldsymbol{\nu}=(\boldsymbol{\nu}_1,\ldots,\boldsymbol{\nu}_{n-1})$ and $i\neq j,\ i,j=1,\ldots,n-1$.

Definition 1.1. We say that $(\boldsymbol{\gamma},\boldsymbol{\nu}):I\to\mathbb{R}^n\times V_{n,n-1}$ is a framed curve if $\dot{\boldsymbol{\gamma}}(t)\cdot\boldsymbol{\nu}_i(t)=0$ for all $t\in I$ and $i=1,\ldots,n-1$.

Definition 1.2. We say that $\boldsymbol{\gamma}:I\to\mathbb{R}^n$ is a framed base curve if there exists $\boldsymbol{\nu}:I\to V_{n,n-1}$ such that $(\boldsymbol{\gamma},\boldsymbol{\nu})$ is a framed curve.

- **Moving frame along $\boldsymbol{\gamma}(t)$**:

$$\{\boldsymbol{\nu}(t),\boldsymbol{\mu}(t)\},$$

where $\boldsymbol{\mu}(t)=\boldsymbol{\nu}_1(t)\times\cdots\times\boldsymbol{\nu}_{n-1}(t)$.

- **Frenet-Serret type formula**:

$$\begin{pmatrix}\dot{\boldsymbol{\nu}}(t)\\ \dot{\boldsymbol{\mu}}(t)\end{pmatrix}=A(t)\begin{pmatrix}\boldsymbol{\nu}(t)\\ \boldsymbol{\mu}(t)\end{pmatrix},$$

where $A(t)=(\alpha_{ij}(t))\in\mathfrak{o}(n)$ for $i.j=1,\ldots,n$. Moreover, there exists a smooth function $\alpha:I\to\mathbb{R}$ such that

$$\dot{\boldsymbol{\gamma}}(t)=\alpha(t)\boldsymbol{\mu}(t).$$

We call the functions $(\alpha_{ij}(t),\alpha(t))$ the **curvature of the framed curve** (with respect to the parameter t).

Definition 1.3. Let $(\boldsymbol{\gamma},\boldsymbol{\nu})$ and $(\widetilde{\boldsymbol{\gamma}},\widetilde{\boldsymbol{\nu}}):I\to\mathbb{R}^n\times V_{n,n-1}$ be framed curves. We say that $(\boldsymbol{\gamma},\boldsymbol{\nu})$ and $(\widetilde{\boldsymbol{\gamma}},\widetilde{\boldsymbol{\nu}})$ are (positive) congruent as framed curves if there exists a matrix $X\in SO(n)$ and a constant vector $\boldsymbol{x}\in\mathbb{R}^n$ such that

$$\widetilde{\boldsymbol{\gamma}}(t)=X(\boldsymbol{\gamma}(t))+\boldsymbol{x},\ \widetilde{\boldsymbol{\nu}}(t)=A(\boldsymbol{\nu}(t))$$

for all $t\in I$.

Then we have the following the existence and the uniqueness theorems similar to the cases of regular space curves.

Theorem 1.4 (The Existence Theorem, [1]). Let $(\alpha_{ij},\alpha):I\to\mathfrak{o}(n)\times\mathbb{R}$ be a smooth mapping. There exists a framed curve $(\boldsymbol{\gamma},\boldsymbol{\nu}):I\to\mathbb{R}^n\times V_{n,n-1}$ whose associated curvature is (α_{ij},α).

Theorem 1.5 (The Uniqueness Theorem, [1]). Let $(\boldsymbol{\gamma},\boldsymbol{\nu})$ and $(\widetilde{\boldsymbol{\gamma}},\widetilde{\boldsymbol{\nu}}):I\to\mathbb{R}^n\times V_{n,n-1}$ be framed curves whose curvatures (α_{ij},α) and $(\widetilde{\alpha}_{ij},\widetilde{\alpha})$ coincide. Then $(\boldsymbol{\gamma},\boldsymbol{\nu})$ and $(\widetilde{\boldsymbol{\gamma}},\widetilde{\boldsymbol{\nu}})$ are congruent as framed curves.

By using the theory of framed curve, we can consider differential geometry of curves with singular points.

2 Framed curves in $\mathbb{R}^3\times V_{3,2}$

Let $(\boldsymbol{\gamma},\boldsymbol{\nu}_1,\boldsymbol{\nu}_2):I\to\mathbb{R}^3\times V_{3,2}$ be a framed curve and $\boldsymbol{\mu}(t)=\boldsymbol{\nu}_1(t)\times\boldsymbol{\nu}_2(t)$. The Frenet-Serret type formula is given by

$$\begin{pmatrix}\dot{\boldsymbol{\nu}}_1(t)\\ \dot{\boldsymbol{\nu}}_2(t)\\ \dot{\boldsymbol{\mu}}(t)\end{pmatrix}=\begin{pmatrix}0 & \ell(t) & m(t)\\ -\ell(t) & 0 & n(t)\\ -m(t) & -n(t) & 0\end{pmatrix}\begin{pmatrix}\boldsymbol{\nu}_1(t)\\ \boldsymbol{\nu}_2(t)\\ \boldsymbol{\mu}(t)\end{pmatrix},$$

where $\ell(t)=\dot{\boldsymbol{\nu}}_1(t)\cdot\boldsymbol{\nu}_2(t)$, $m(t)=\dot{\boldsymbol{\nu}}_1(t)\cdot\boldsymbol{\mu}(t)$ and $n(t)=\dot{\boldsymbol{\nu}}_2(t)\cdot\boldsymbol{\mu}(t)$. Moreover, there exists a smooth function $\alpha:I\to\mathbb{R}$ such that

$$\dot{\boldsymbol{\gamma}}(t)=\alpha(t)\boldsymbol{\mu}(t).$$

Example 2.1. Let $\boldsymbol{\gamma}:I\to\mathbb{R}^3$ be a regular curve with linear independent condition, namely, $\dot{\boldsymbol{\gamma}}(t)$ and $\ddot{\boldsymbol{\gamma}}(t)$ are linear independent for all $t\in I$. Then $\boldsymbol{\gamma}:I\to\mathbb{R}^3$ is a framed base curve. This means that framed base curves are generalization of regular curves.

Proposition 2.2 ([1])**. Let $\boldsymbol{\gamma}:(I,t_0)\to\mathbb{R}^3$ be an analytic germ. Then $\boldsymbol{\gamma}$ is a framed base curve. Namely, there exists a germ $(\boldsymbol{\nu}_1,\boldsymbol{\nu}_2):(I,t_0)\to V_{3,2}$ such that $(\boldsymbol{\gamma},\boldsymbol{\nu}_1,\boldsymbol{\nu}_2):(I,t_0)\to\mathbb{R}^3\times V_{3,2}$ is a framed curve.**

In addition, we investigated the following properties of general framed curves in [1, 2]:

- **Frame change** (rotated frame and reflected frame).
- **Projection to plane and Legendre curve**.
- **Contact between framed curves**. etc...

In general, the moving frame of a framed curve does not have geometric meaning. However, we can consider a moving frame with geometric meaning under a certain condition.

Definition 2.3. We say that $\boldsymbol{\gamma}:I\to\mathbb{R}^3$ is a Frenet-type framed base curve if there exist a regular spherical curve $\boldsymbol{t}:I\to S^2$ and a function $\alpha:I\to\mathbb{R}$ such that $\dot{\boldsymbol{\gamma}}(t)=\alpha(t)\boldsymbol{t}(t)$ for all $t\in I$. Then we call $\boldsymbol{t}(t)$ a unit tangent vector and $\alpha(t)$ a speed function of $\boldsymbol{\gamma}(t)$.

- **Frenet-type frame**:

$$\{\boldsymbol{t}(t),\boldsymbol{n}(t),\boldsymbol{b}(t)\},$$

where $\boldsymbol{n}(t)=\dot{\boldsymbol{t}}(t)/\|\dot{\boldsymbol{t}}(t)\|$ and $\boldsymbol{b}(t)=\boldsymbol{t}(t)\times\boldsymbol{n}(t)$.

- **Frenet-Serret type formula**:

$$\begin{pmatrix}\dot{\boldsymbol{t}}(t)\\ \dot{\boldsymbol{n}}(t)\\ \dot{\boldsymbol{b}}(t)\end{pmatrix}=\begin{pmatrix}0 & \kappa(t) & 0\\ -\kappa(t) & 0 & \tau(t)\\ 0 & -\tau(t) & 0\end{pmatrix}\begin{pmatrix}\boldsymbol{t}(t)\\ \boldsymbol{n}(t)\\ \boldsymbol{b}(t)\end{pmatrix},$$

where

$$\kappa(t)=\|\dot{\boldsymbol{t}}(t)\|,\ \tau(t)=\frac{\det(\boldsymbol{t}(t),\dot{\boldsymbol{t}}(t),\ddot{\boldsymbol{t}}(t))}{\|\dot{\boldsymbol{t}}(t)\|^2}.$$

We call $\kappa(t)$ the **curvature** and $\tau(t)$ the **torsion** of $\boldsymbol{\gamma}$ with $\boldsymbol{t}$.

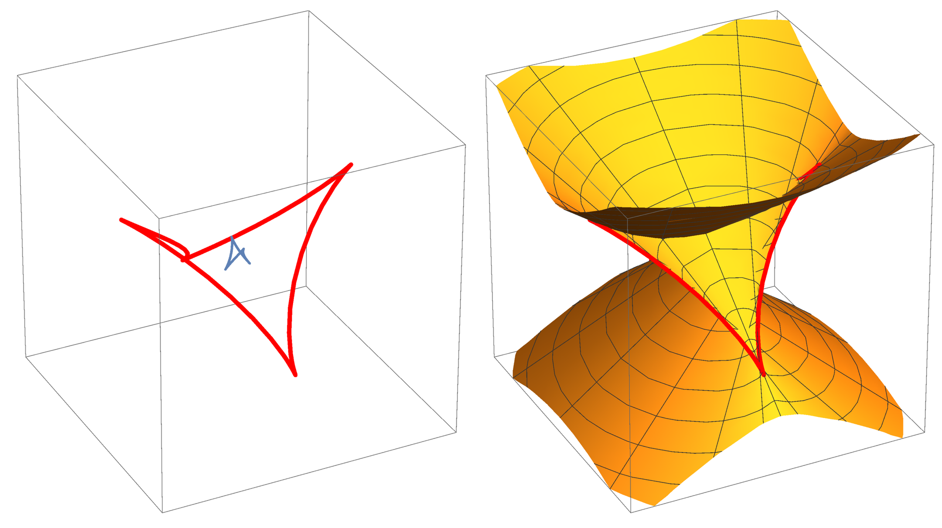
Frenet-type frame is natural generalization of Frenet-frame for a regular space curve with linear independent condition. We can easily check that $(\boldsymbol{\gamma},\boldsymbol{n},\boldsymbol{b}):I\to\mathbb{R}^3\times V_{3,2}$ is a framed curve, so that we can apply the theory of framed curves.

We cannot define the rectifying developable surface in §3 without the notion of Frenet-type frame.

3 Applications

In this section, we roughly consider applications by using figures of specific examples.

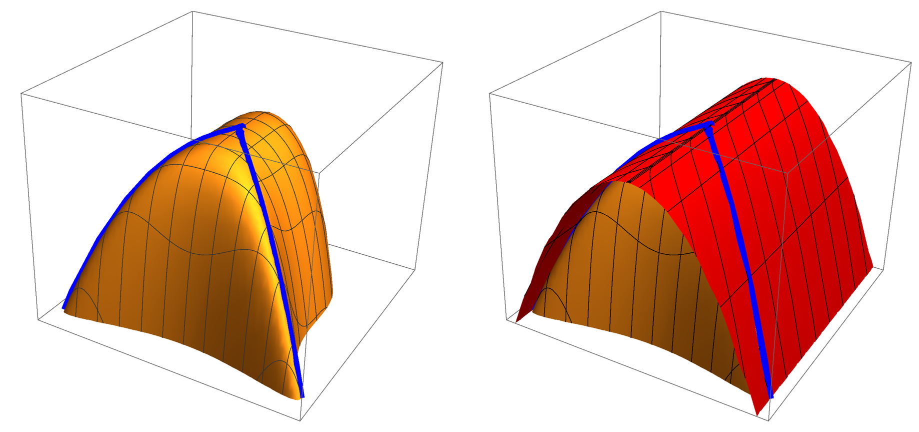
- **Evolute and Focal developable surface of Frenet-type framed base curve** (cf. [2]):



(a) (b)

Figure 1: (a) is the the astroid (blue curve) and its evolute (red curve). (b) is the focal developable surface of the astroid.

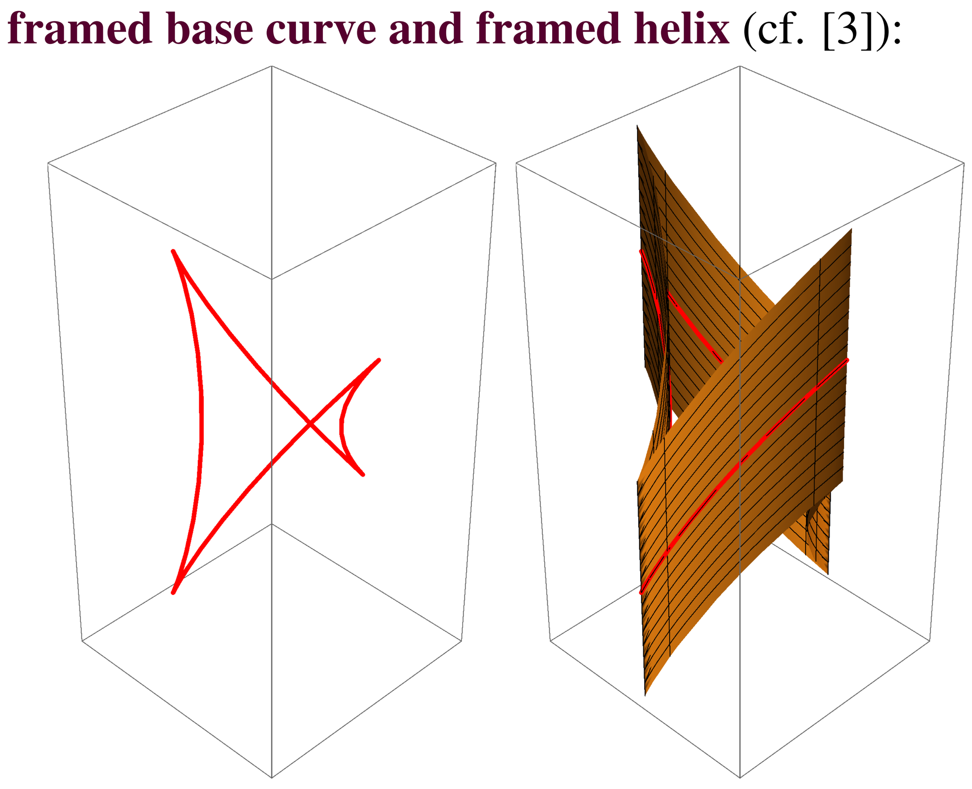
- **Osculating developable surface of frontal curve on surface** (cf. [4]):



(c) (d)

Figure 2: (c) is a regular surface and a frontal curve on the surface. (d) is osculating developable surface along the curve. In this case, the curve is a contour generator with respect to an orthogonal projection.

- **Rectifying developable surface of Frenet-type framed base curve and framed helix** (cf. [3]):



(e) (f)

Figure 3: (e) is the astroid. We can easily check that the astroid is a framed helix. (f) is the rectifying developable surface of the curve. In this case, the surface is a cylinder.

In my presentation, I'll talk about the geometric meaning of Frenet-type frame, the detail of the rectifying developable surface and framed helix.

References

[1] S. Honda and M. Takahashi, Framed curves in the Euclidean space, *Adv. Geom.*, 16:265–276, 2016.

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