Developable surfaces along framed base curves

Shun'ichi Honda

Hokkaido University

June 26-30, 2017 @ Boston University

Introduction

Developable surfaces along framed base curves

We briefly review two keywords !!

Introduction

Developable surfaces along framed base curves

Ruled surfaces with $K = 0$

$$F_{(\gamma,\xi)}(t,u) = \gamma(t) + u\xi(t)$$

$\gamma : I \to \mathbb{R}^3$: **Base curve**

$\xi : I \to \mathbb{R}^3 \setminus \{0\}$: **Director curve**

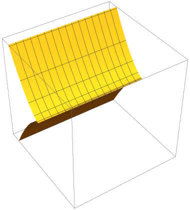
$\gamma(t_0) + u\xi(t_0)$: **Ruling (Straight line)**

Introduction

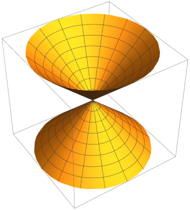
- **The classification of developable surfaces**:

$F_{(\gamma,\xi)}$:	**Cylinder**	$\dot{\boldsymbol{\xi}}(t) = \mathbf{0}$
$F_{(\gamma,\xi)}$:	**Cone**	$\dot{\boldsymbol{\sigma}}(t) = \mathbf{0}$
$F_{(\gamma,\xi)}$:	**Tangent developable**	$\dot{\boldsymbol{\sigma}}(t) \neq \mathbf{0}$

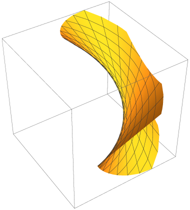
$$\text{Here, } \boldsymbol{\sigma}(t) = \boldsymbol{\gamma}(t) - \frac{\dot{\boldsymbol{\gamma}}(t)\cdot\dot{\boldsymbol{\xi}}(t)}{\dot{\boldsymbol{\xi}}(t)\cdot\dot{\boldsymbol{\xi}}(t)}\boldsymbol{\xi}(t) \ (\text{when, } \ \delta(t) \neq \mathbf{0}).$$



Cylinder



Cone



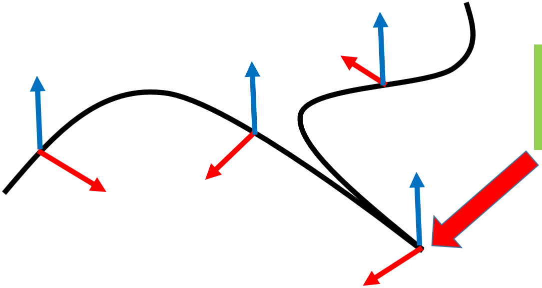
Tangent developable

Introduction

Developable surfaces along framed base curves

Space curve γ with a moving frame

(γ may have singular points)



A singular point
($\dot{\gamma}(t) = 0$)

• **Developable surfaces associated to regular space curves**:

- Tangent developable surfaces:
  There are many articles
- Focal developable surfaces:
  There are many articles
- **Rectifying developable surfaces**:
  [Izumiya-Katsumi-Yamasaki]

• **Developable surfaces associated to "singular" space curves**:

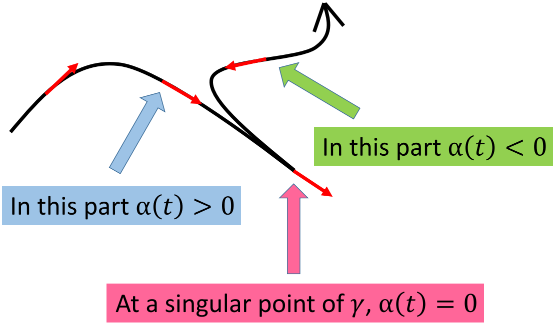
- Tangent developable surfaces:
  [Ishikawa]
- Focal developable surfaces:
  [H-Takahashi, submitted]
- **Rectifying developable surfaces**:
  [H, accepted]

Frenet type framed base curve

Definition (Frenet type framed base curve)

We say that $\gamma : I \to \mathbb{R}^3$ is a **Frenet type framed base curve** if ${}^\exists \boldsymbol{t} : I \to S^2$: **Regular** spherical curve and ${}^\exists \alpha : I \to \mathbb{R}$ s.t.

$$\dot{\gamma}(t) = \frac{d\gamma}{dt}(t) = \alpha(t)\boldsymbol{t}(t).$$



- **Frenet type frame**:

$$\left\{ \boldsymbol{t}(t), \boldsymbol{n}(t) = \frac{\dot{\boldsymbol{t}}(t)}{\|\dot{\boldsymbol{t}}(t)\|}, \boldsymbol{b}(t) = \boldsymbol{t}(t) \times \boldsymbol{n}(t) \right\}.$$

- **Frenet-Serret type formula**:

$$\begin{pmatrix} \dot{\boldsymbol{t}}(t) \\ \dot{\boldsymbol{n}}(t) \\ \dot{\boldsymbol{b}}(t) \end{pmatrix} = \begin{pmatrix} 0 & \kappa(t) & 0 \\ -\kappa(t) & 0 & \tau(t) \\ 0 & -\tau(t) & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{t}(t) \\ \boldsymbol{n}(t) \\ \boldsymbol{b}(t) \end{pmatrix}.$$

- **Invariants of Frenet type framed base curve**:

$$\begin{aligned} \textbf{Speed} &: \quad \alpha(t), \\ \textbf{Curvature} &: \quad \kappa(t) = \|\dot{\boldsymbol{t}}(t)\|, \\ \textbf{Torsion} &: \quad \tau(t) = \frac{\det\left(\boldsymbol{t}(t), \dot{\boldsymbol{t}}(t), \ddot{\boldsymbol{t}}(t)\right)}{\|\dot{\boldsymbol{t}}(t)\|^2}. \end{aligned}$$

Remark

$(\alpha(t), \kappa(t), \tau(t))$ is an invariant of a Frenet type framed base curve, up to congruence. (cf. [H-Takahashi, 2016] and **my poster**)

- **The spherical Darboux type vector**:

$$\overline{\mathcal{D}}(t) = \frac{\tau(t)\boldsymbol{t}(t) + \kappa(t)\boldsymbol{b}(t)}{\sqrt{\kappa^2(t) + \tau^2(t)}}.$$

Remark

$\overline{\mathcal{D}}(t)$ is a spherical dual of $\boldsymbol{n}(t)$.

Rectifying developable surface

Definition (Rectifying developable surface)

We define the **rectifying developable surface** $\mathcal{RD}_\gamma : I \times \mathbb{R} \to \mathbb{R}^3$ by

$$\mathcal{RD}_\gamma(t,u) = \boldsymbol{\gamma}(t) + u\overline{D}(t) = \boldsymbol{\gamma}(t) + u\frac{\tau(t)\boldsymbol{t}(t) + \kappa(t)\boldsymbol{b}(t)}{\sqrt{\kappa^2(t) + \tau^2(t)}}.$$

$(t_0, u_0) \in I \times \mathbb{R}$ is a singular point of $\mathcal{RD}_\gamma$ if and only if

$$\frac{\alpha(t_0)\kappa(t_0)}{\sqrt{\kappa^2(t) + \tau^2(t)}} + u_0\frac{\kappa(t_0)\dot{\tau}(t_0) - \dot{\kappa}(t_0)\tau(t_0)}{\kappa^2(t_0) + \tau^2(t_0)} = 0.$$

Remark

In regular parts of $\mathcal{RD}_\gamma$, γ are geodesics on $\mathcal{RD}_\gamma$.

Theorem A-(1)

$\mathcal{RD}_\gamma$ is locally diffeomorphic to the cuspidal edge $\boldsymbol{ce}$ at (t_0, u_0) if and only if

(i) $\delta(t_0) \neq 0$, $\sigma(t_0) \neq 0$ and

$$u_0 = -\frac{\alpha(t_0)\kappa(t_0)}{\delta(t_0)\sqrt{\kappa^2(t_0) + \tau^2(t_0)}},$$

or

(ii) $\delta(t_0) = \alpha(t_0) = 0$, $\dot{\delta}(t_0) \neq 0$ and

$$u_0 \neq -\dot{\alpha}(t_0)\kappa(t_0)\frac{\sqrt{\kappa^2(t_0) + \tau^2(t_0)}}{\kappa(t_0)\ddot{\tau}(t_0) - \ddot{\kappa}(t_0)\tau(t_0)},$$

or

(iii) $\delta(t_0) = \alpha(t_0) = 0$ and $\dot{\alpha}(t_0) \neq 0$.

Theorem A-(2)

$\mathcal{RD}_\gamma$ is locally diffeomorphic to the swallowtail $\boldsymbol{sw}$ at (t_0, u_0) if and only if $\delta(t_0) \neq 0$, $\sigma(t_0) = 0$, $\dot{\sigma}(t_0) \neq 0$ and

$$u_0 = -\frac{\alpha(t_0)\kappa(t_0)}{\delta(t_0)\sqrt{\kappa^2(t_0) + \tau^2(t_0)}}.$$

Theorem A-(3)

$\mathcal{RD}_\gamma$ is never locally diffeomorphic to the cuspidal cross cap $\boldsymbol{ccc}$.

Here,

$$\delta(t) = \frac{\kappa(t)\dot{\tau}(t) - \dot{\kappa}(t)\tau(t)}{\kappa^2(t) + \tau^2(t)},$$

$$\sigma(t) = \frac{\alpha(t)\tau(t)}{\sqrt{\kappa^2(t) + \tau^2(t)}} - \frac{d}{dt}\left(\frac{\alpha(t)\kappa(t)}{\delta(t)\sqrt{\kappa^2(t) + \tau^2(t)}}\right) \quad (\delta(t) \neq 0).$$

Framed helix

Definition (Framed helix)

Let $\gamma : I \to \mathbb{R}^3$ be a **Frenet type framed base curve**.
We say that γ is a **framed helix** if ${}^{\exists}\boldsymbol{v} \in S^2$ and ${}^{\exists}c \in \mathbb{R}$ s.t. $\boldsymbol{t}(t) \cdot \boldsymbol{v} = c$ for ${}^{\forall}t \in I$.

Theorem B-

(1) The following are equivarent:
 (i) $\mathcal{RD}_\gamma$ is a cylinder,
 (ii) $\delta(t) = 0$ for all $t \in I$,
 (iii) γ is a framed helix.

(2)Suppose that $\delta(t) \neq 0$, then the following are equivarent:
 (i) $\mathcal{RD}_\gamma$ is a cone,
 (ii) $\sigma(t) = 0$ for all $t \in I$.

Example (The astroid)

$$\begin{cases} \boldsymbol{\gamma} : [0, 2\pi) \to \mathbb{R}^3, & \boldsymbol{\gamma}(t) = \left(\cos^3 t, \sin^3 t, \cos 2t\right), \\ \alpha : [0, 2\pi) \to \mathbb{R}, & \alpha(t) = 5 \cos t \sin t, \\ \boldsymbol{t} : [0, 2\pi) \to S^2, & \boldsymbol{t}(t) = (1/5)\left(-3\cos t, \sin t, -4\right). \end{cases}$$

By a direct calculation, we have $\kappa(t) = 3/5$, $\tau(t) = 4/5$ and $\delta(t) = 0$. Hence $\boldsymbol{\gamma}$ is a framed helix and $\mathcal{RD}_{\boldsymbol{\gamma}}$ is a cylinder.

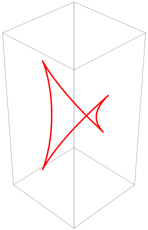


Figure: The image of $\boldsymbol{\gamma}$

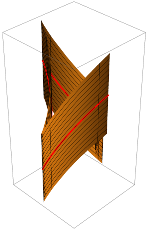
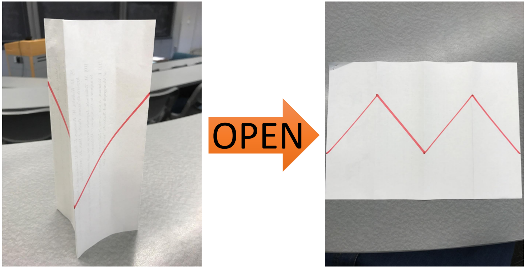


Figure: The image of $\mathcal{RD}_{\boldsymbol{\gamma}}$

This means that, in the case when we open the surface as "origami", we have the following pictures:



• Main References:

[1] S. Honda and M. Takahashi, Framed curves in the Euclidean space, *Adv. Geom.*, 16:265-276, 2016.

[2] S. Honda, **Rectifying developable developable surfaces of framed base curves and framed helices**, accepted.