The $\mathrm{SL}(n)$-representation of fundamental groups and related topics

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Abstract

In this poster, we discuss the higher Teichmüller space. In particular, the work of Bonahon and Dreyer about a coordinate of the higher Teichmüller theory is explained.

Higher Teichmüller Space

In this section, we define the higher Teichmüller space which is a connected component of the $\mathrm{PSL}_n(\mathbb{R})$-representation space.

Notations

- S : an orientable surface with a negative Euler characteristic number.
- $\mathcal{R}^n = \mathrm{Hom}(\pi_1(S), \mathrm{PSL}_n(\mathbb{R}))$ with the compact open topology.
- $\mathcal{R}^2_{geom}$: a set of discrete, faithful representations of $\pi_1(S)$ to $\mathrm{PSL}_2(\mathbb{R})$.

Recall that the space $\mathcal{R}^2_{geom}$ gives the (classical) Teichmüller space in the hyperbolic geometry. If we denote the Teichmüller space as $\mathcal{T}(S)$, then the following identification holds,

$$\mathcal{T}(S) = \mathcal{R}^2_{geom}/\mathrm{PSL}_2(\mathbb{R}).$$

To define the higher Teichmüller space, We use the next fact which is well known in the representation theory:

Proposition

There exists a unique irreducible $\mathrm{PSL}_n(\mathbb{R})$-representation of $\mathrm{PSL}_2(\mathbb{R})$:

$$\iota_n : \mathrm{PSL}_2(\mathbb{R}) \to \mathrm{PSL}_n(\mathbb{R}).$$

By using the induced map $(\iota_n)_* : \mathcal{R}^2 \to \mathcal{R}^n$, we take a connected component $\mathcal{R}^n_{geom}$ in $\mathcal{R}^n$ which contains a subset $(\iota_n)_*(\mathcal{R}^2_{geom})$.

Definition

The higher Teichmüller space for a surface S is the space defined by

$$\mathrm{Hit}_n(S) = \mathcal{R}^n_{geom}/\mathrm{PSL}_n(\mathbb{R}).$$

In this poster we use the following notations.

- $(\iota_n)_*(\mathcal{R}^2_{geom})/\mathrm{PSL}_n(\mathbb{R})$: **Fuchsian Locus**.
- $\rho \in \mathrm{Hit}_n(S)$: **Hitchin representation**.

Remarks

1. The higher Teichmüller space was originally defined by Hitchin ([4]). Therefore this space is often called the Hitchin component.
2. When S has nonempty boundary, we define $\mathcal{R}^n_{geom}$ as a component in the purely-loxodromic representation space.

Anosov Property of Hitchin Representation

Labourie studied the geometric properties of the Hitchin representation focusing on some geometric structure which he called the Anosov structure ([6]). In his article he defined so-called the **Anosov representation** and proved the next theorem.

Theorem(Labourie '06)

- Hitchin representations are Anosov.
- In particular, Hitchin representations are discrete and faithful.

The Anosov representation is the holonomy representation associated to the Anosov structure and is defined by using the dynamical theory and the Lie group theory.

The concept of the Anosov representation is now extended to representations of the general word hyperbolic groups by Guichard-Wienhard ([3]). Moreover this representation is studied in the context of the geometric group theory ([5]). We can also study the higher Teichmüller space algebraically by the cluster algebra theory and the positive representation theory ([2]).

The following theorem is an important fact which is often used to study the Anosov representation.

Theorem(Labourie '06)

For any Hitchin representation $\rho : \pi_1(S) \to \mathrm{PSL}_n(\mathbb{R})$, there exists a unique ρ-equivariant continuous map $\mathcal{F}_\rho : \partial_\infty \pi_1(S) \to Flag(\mathbb{R}^n)$.

The symbols $\partial_\infty \pi_1(S)$ and $Flag(\mathbb{R}^n)$ respectively mean the Gromov boundary of $\pi_1(S)$ and the space of flags which are nested sequences of subspaces in $\mathbb{R}^n$. The map $\mathcal{F}_\rho$ is called a **limit curve** or a **flag curve** of ρ.

Bonahon-Dreyer's Coordinate

We introduce a coordinate on the higher Teichmüller space defined by Bonahon and Dreyer ([1]). To explain this, we need some projective invariants.

- $(E, F, G) \in Flag(\mathbb{R}^n)^3$: a generic triple of flags

$$T_{abc}(E,F,G) = \frac{e^{(a+1)} \wedge f^{(b)} \wedge g^{(c-1)}}{e^{(a-1)} \wedge f^{(b)} \wedge g^{(c+1)}} \frac{e^{(a)} \wedge f^{(b-1)} \wedge g^{(c+1)}}{e^{(a)} \wedge f^{(b+1)} \wedge g^{(c-1)}} \frac{e^{(a-1)} \wedge f^{(b+1)} \wedge g^{(c)}}{e^{(a+1)} \wedge f^{(b-1)} \wedge g^{(c)}}.$$

- $(E, F, G, G') \in Flag(\mathbb{R}^n)^4$: a generic quadruple of flags

$$D_a(E,F,G,G') = -\frac{e^{(a)} \wedge f^{(n-a-1)} \wedge g^{(1)}}{e^{(a)} \wedge f^{(n-a-1)} \wedge g'^{(1)}} \frac{e^{(a-1)} \wedge f^{(n-a)} \wedge g'^{(1)}}{e^{(a-1)} \wedge f^{(n-a)} \wedge g^{(1)}}.$$

The first invariant is called the **triple ratio** and the second is called the **double ratio**. In the definition of the triple ratio, (a, b, c) is a triple of integers which satisfy $a, b, c \geq 1, a + b + c = n$, and of the double ratio, $a = 1, \cdots, n-1$. A generic pair of flags is a pair of flags which intersect transversally each other.

The construction of the Bonahon-Dreyer coordinate.

1. We fix a maximal geodesic lamination λ on the surface S. Suppose that closed leaves $g_i \in \lambda$ have transverse arcs k_i.
2. Orient leaves of this lamination. We can choose these orientation independently for each leaf.
3. Fix lifts of components T_i of $S \setminus \lambda$, biinfinite leaves h_i, closed leaves g_i of λ, and transverse arcs k_i. We denote these lifts as $\widetilde{T}_i, \widetilde{h}_i, \widetilde{g}_i, \widetilde{k}_i$.
4. For a Hitchin representation ρ, take a flag curve $\mathcal{F}_\rho$ and give an invariant for $\widetilde{T}_i, \widetilde{h}_i, \widetilde{g}_i$ as follows.

Triangle Invariant and Shearing Invariant

$$\tau^\rho_{abc}(T_i, v_i) = \log T_{abc}(\mathcal{F}_\rho(\tilde{v}_i), \mathcal{F}_\rho(\tilde{v}_i'), \mathcal{F}_\rho(\tilde{v}_i'')).$$
$$\sigma^\rho_a(h_i) = \log D_a(\mathcal{F}_\rho(x), \mathcal{F}_\rho(y), \mathcal{F}_\rho(z), \mathcal{F}_\rho(z')).$$
$$\sigma^\rho_a(g_i) = \log D_a(\mathcal{F}_\rho(x), \mathcal{F}_\rho(y), \mathcal{F}_\rho(z), \mathcal{F}_\rho(z')).$$

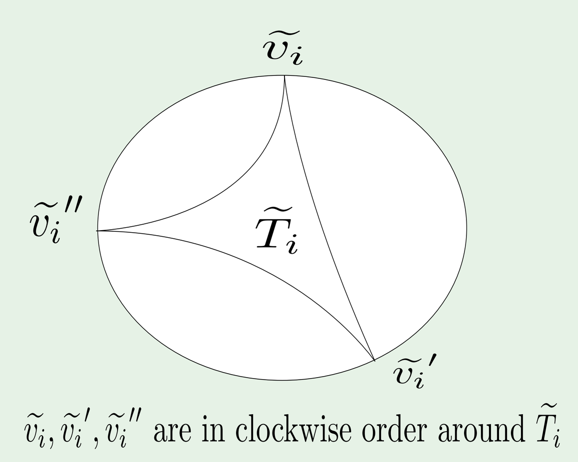

Figure 1: A lift of a triangle T_i.

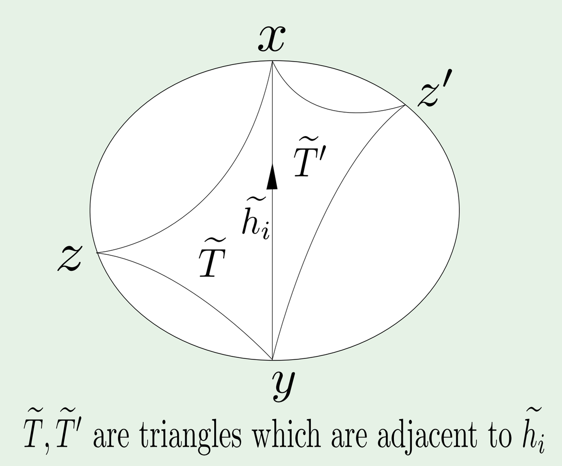

Figure 2: A lift of a biinfinite leaf h_i.

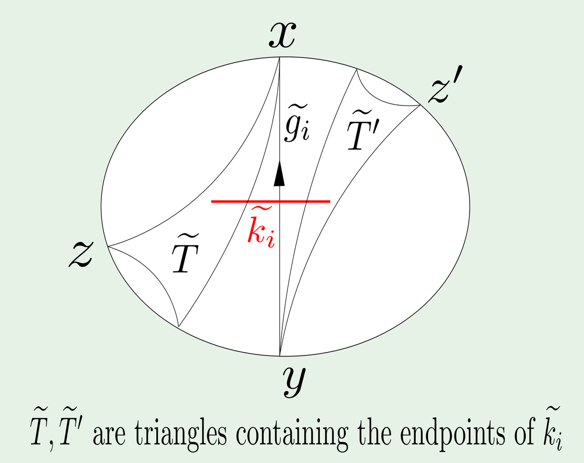

Figure 3: A lift of a closed leaf g_i.

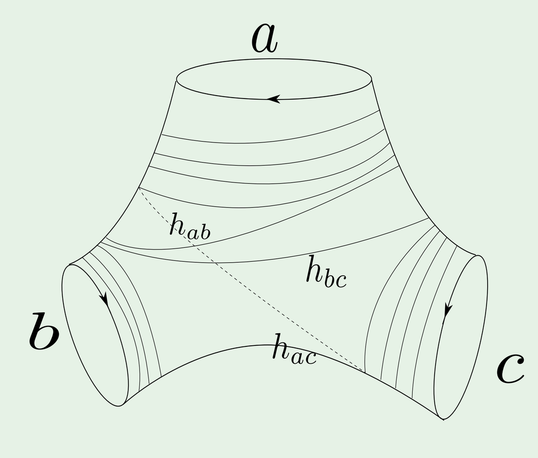

Figure 4: A lamination on a pants.

The invariant τ_{abc} and σ_a are called the **triangle invariant** and the **shearing invariant**. Bonahon and Dreyer used these invariants to parametrize the higher Teichmüller space.

Theorem(Bonahon-Dreyer '14)

A map induced by the triangle, shearing invariant is a homeomorphism from the higher Teichmüller space onto the interior of a polytope $\overline{\mathcal{P}}$.

$$\Phi : \mathrm{Hit}_n(S) \to \mathcal{P} : \rho \mapsto (\tau^\rho_{abc}, \sigma^\rho_a).$$

Result

Recall that the Fuchsian locus is a locus in the higher Teichmüller space which appears from the Teichmüller space $\mathcal{T}(S)$. We can explicitly describe the Fuchsian locus under the Bonahon-Dreyer's coordinate for a pants. The following is a computational result of the case $\mathrm{PSL}_3(\mathbb{R})$.

Result (I.)

Fix a lamination on a pants S as a figure 4. Then the $\mathrm{PSL}_3(\mathbb{R})$-Fuchsian Locus is described as follows. Here, α, β, γ are variables determined by the hyperbolic structure of the pants.

$$\tau_{111}(T_0, v = \infty) = 0, \quad \tau_{111}(T_1, v = \infty) = 0.$$
$$\sigma_1(h_{ab}) = \log(\frac{1}{\beta\gamma}), \quad \sigma_2(h_{ab}) = \log(\frac{1}{\beta\gamma}).$$
$$\sigma_1(h_{ac}) = \log(\alpha^2\beta\gamma), \quad \sigma_2(h_{ac}) = \log(\alpha^2\beta\gamma).$$
$$\sigma_1(h_{bc}) = \log(\frac{\beta}{\gamma}), \quad \sigma_2(h_{bc}) = \log(\frac{\beta}{\gamma}).$$

[1] F. Bonahon and G. Dreyer, *Parameterizing Hitchin components*, Duke Math. J. *163*(2014), no. 15, 2935-2975.

[2] V. Fock and A. Goncharov, *Moduli spaces of local systems and higher Teichmüller theory*, Publ. Math. Inst. Hautes Études Sci. No. **103**(2006) 1-211.

[3] O. Guichard and A. Wienhard, *Anosov representations: domains of discontinuity and application*, Invent. Math. No. **190**(2012) 357-438.

[4] N. Hitchin, *Lie groups and Teichmüller space*, Topology **31** (1992), no.3, 449-473.

[5] M. Kapovich, B. Leeb and J. Porti, *Morse action of discrete groups on symmetric space*, arXiv: 1403.7671

[6] F. Labourie, *Anosov flows, surface groups and curves in projective space*, Invent. Math. **165**(2006), no. 1, 51-114.