The $\mathrm{SL}(n)$-representation of fundamental groups and related topics

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Teichmüller Space

The Teichmüller Space is a moduli space of some structures of surfaces, for example, **complex structure**, **conformal structure**, and **hyperbolic structure**.

- S: a closed connected orientable surface with a negative Euler characteristic number.
- $\mathcal{M}(S)$: the set of Riemannian metrics on S.
- $\mathrm{Diff}_0(S)$: the identity component of the diffeomorphism group of S.

Definition(Teichmüller Space)

$$\mathcal{T}(S) = \mathcal{M}(S)/\mathrm{Diff}_0(S).$$

This set consists of the Riemannian metrics with the negative constant curvature, which are called the **hyperbolic structures** of S.

Hyperbolic Structures and Representations

- g: a hyperbolic structure of S.
- $\widetilde{S}$: the universal covering of S.
- $\mathbb{H}^2$: the upper half space with the metric $ds^2 = \frac{dx^2+dy^2}{y^2}$.

Then there exists an isometric identification called the **developing map**.

$$Dev_g : \widetilde{S} \to \mathbb{H}^2.$$

Then we can take a representation

$$\rho_g : \pi_1(S) \to \mathrm{Isom}^+(\mathbb{H}^2) \cong \mathrm{PSL}_2(\mathbb{R})$$

so that Dev_g is ρ_g- equivariant.

- Such a representation is called the **holonomy representation**.
- These holonomy representation are discrete and faithful.
- The surface with the Riemannian metric g is reconstructed by the quotient $\mathbb{H}^2/\rho_g(\pi_1(S))$.

Hyperbolic Structures and Representations

The correspondence between hyperbolic structures and representations of the fundamental group gives the following identification.

- $\mathcal{R}^2 = \mathrm{Hom}(\pi_1(S), \mathrm{PSL}_2(\mathbb{R}))$ is the space of representations with the compact open topology.
- $\mathcal{R}^2_{geom}$ is the subset of discrete and faithful representations in $\mathcal{R}^2$.
- $\mathrm{PSL}_2(\mathbb{R})$ acts on $\mathcal{R}^2_{geom}$ by conjugation.

Proposition

The following map is bijective.

$$Hol : \mathcal{T}(S) \to \mathcal{R}^2_{geom}/\mathrm{PSL}_2(\mathbb{R}) : g \mapsto \rho_g.$$

We remark that $\mathcal{R}^2_{geom}$ is a connected component of $\mathcal{R}^2$ (Goldman '88). This component is often called the **Teichmüller component**.

Definition of Higher Teichmüller Space

The higher Teichmüller space is a generalization of the Teichmüller space in the sense of the representation space.

- $\mathcal{R}^n = \mathrm{Hom}(\pi_1(S), \mathrm{PSL}_n(\mathbb{R}))$ with the compact open topology.
- $\mathcal{R}^2_{geom}$ is the Teichmüller component that is the set of discrete and faithful representations.

Proposition(Canonical Irreducible Representation)

There exists a unique irreducible $\mathrm{PSL}_n(\mathbb{R})$-representation of $\mathrm{PSL}_2(\mathbb{R})$ up to conjugation.

$$\iota_n : \mathrm{PSL}_2(\mathbb{R}) \to \mathrm{PSL}_n(\mathbb{R}).$$

This representation induces the following correspondence $(\iota_n)_*$:

$$(\iota_n)_* : \mathcal{R}^2 \ni \rho \to \iota_n \circ \rho \in \mathcal{R}^n.$$

Definition of Higher Teichmüller Space

- $\mathcal{R}^n = \mathrm{Hom}(\pi_1(S), \mathrm{PSL}_n(\mathbb{R}))$.
- $\iota_n : \mathrm{PSL}_2(\mathbb{R}) \to \mathrm{PSL}_n(\mathbb{R})$.
- $(\iota_n)_* : \mathcal{R}^2 \ni \rho \to \iota_n \circ \rho \in \mathcal{R}^n$.

We can consider a component $\mathcal{R}^n_{geom}$ in $\mathcal{R}^n$ containing $(\iota_n)_*(\mathcal{R}^2_{geom})$. Note that $\mathrm{PSL}_n(\mathbb{R})$ acts on $\mathcal{R}^n_{geom}$ by conjugation.

Definition(Higher Teichmüller Space)

The higher Teichmüller space is the quotient space

$$\mathrm{Hit}_n(S) = \mathcal{R}^n_{geom}/\mathrm{PSL}_n(\mathbb{R}).$$

Terminology

- $(\iota_n)_*(\mathcal{R}^2_{geom})/\mathrm{PSL}_n(\mathbb{R})$: **Fuchsian Locus**.
- $\rho \in \mathrm{Hit}_n(S)$: **Hitchin representation**.

Some Works in Higher Teichmüller Theory

1. Hitchin

The higher Teichmüller space is originally defined by Hitchin. Therefore this space is often called the **Hitchin component**. He used the Higgs bundle to study the representation space and showed the following theorem.

Theorem(Hitchin '92)

$\mathrm{Hit}_n(S)$ is homeomorphic to $\mathbb{R}^{(2g-2)(n^2-1)}$.

- However, in his article, he said
  ... Unfortunately, the analytical point of view used for the proofs gives no indication of the geometrical significance of the Teichmüller component. ...

Some Works in Higher Teichmüller Theory

2. Labourie

Labourie studied the geometric properties of the Hitchin representation. He defined the geometric structure, the **Anosov structure** and its holonomy representation, the **Anosov representation**. The insight of this representation gives the following theorem:

Theorem(Labourie '06)

- The Hitchin representations are Anosov.
- The Hitchin representations are discrete and faithful.

- The concept of Anosov representation is extended to the representation of the general word hyperbolic group (Guichard-Wienhard '12).
- We can study the higher Teichmüller theory in the viewpoint of the positive representations (Fock-Goncharov '06).

Some Works in Higher Teichmüller Theory

3. Higher Teichmüller Space and Geometric structures

- $n = 3$: the real projective structure on surfaces (Choi-Goldman '93).
- $n = 4$: convex foliated projective structure of the unit tangent bundle over the surface T^1S (Guichard-Wienhard '08).

The real projective structure is a geometric structure locally modeled on the projective space $\mathbb{RP}^n$.

Theorem(Choi-Goldman '93)

The moduli space of the real projective structure on surfaces agrees with $\mathrm{PSL}_3(\mathbb{R})$-higher Teichmüller space.

More precisely, if $\rho \in \mathrm{Hit}_3(S)$, then there exists an open convex domain Ω in $\mathbb{RP}^2$ such that $S = \Omega/\rho(\pi_1(S))$. Conversely any real projective surface is given by some Hitchin representation.

4. Bonahon-Dreyer

Bonahon-Dreyer defined a coordinate by using the Anosov property of the Hitchin representations and the topological data of surface, the lamination which is a family of simple closed and biinfinite curves on S.

Theorem(Bonahon-Dreyer '14)

λ: the maximal lamination on the surface S.
There exists a homeomorphism from the Hitchin component onto the interior of a convex polytope.

$$\Phi_\lambda : \mathrm{Hit}_n(S) \to \mathcal{P}.$$

They defined two invariants, the **shearing invariant** and the **triangle invariant** by using the Anosov property of the Hitchin representations. This coordinate is introduced in my poster.

Result

Question.

Can we explicitly describe the Fuchsian locus under the Bonahon-Dreyer coordinate ?

Result (I.)

When S is the pants and $n = 3$, we can describe the Fuchsian locus as the following.

$$\tau_{111}(T_0, v = \infty) = 0, \qquad \tau_{111}(T_1, v = \infty) = 0.$$
$$\sigma_1(h_{ab}) = \log(\frac{1}{\beta\gamma}), \qquad \sigma_2(h_{ab}) = \log(\frac{1}{\beta\gamma}).$$
$$\sigma_1(h_{ac}) = \log(\alpha^2\beta\gamma), \qquad \sigma_2(h_{ac}) = \log(\alpha^2\beta\gamma).$$
$$\sigma_1(h_{bc}) = \log(\frac{\beta}{\gamma}), \qquad \sigma_2(h_{bc}) = \log(\frac{\beta}{\gamma}).$$

where $(\alpha, \beta, \gamma) \in \mathbb{R}^3$ is determined by the hyperbolic structures of pants.

Thank you for your attention!!

Reference

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