A fixed-point property of random groups

joint work with Takefumi Kondo and Shin Nayatani

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$\mathcal{H}$: Hilbert space
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Note. We don't assume ρ to be injective etc.

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$\star$ Hyperbolic groups were thought to be flexible. (e.g.,

· Lattices in $SO(n,1)$ and $SU(n,1)$ are not (T).

· Lattices in $Sp(n,1)$ is not so rigid as higher rank lattices.)

Affine action on Hilbert spaces

Def. $\varphi\colon \mathcal{H} \to \mathcal{H}$ is affine if $\varphi(x) = Ax + v$ for some $v \in \mathcal{H}$ and an invertible $A \in \mathbb{B}(\mathcal{H})$, and $\mathrm{Aff}(\mathcal{H})$ denotes the group of affine transformations on $\mathcal{H}$.
$\rho\colon \Gamma \to \mathrm{Aff}(\mathcal{H})$ is uniformly C-Lipschitz (UL(C))
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Thm 1. (I-Kondo-Nayatani) Fix $C \geq 1$. Any UL(C) action of certain random groups on $\mathcal{H}$ has a fixed point.

Plain word model of random groups

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Fix $c > 1$. For $0 < d < 1$ we set

$$P(m, \ell, d) = \left\{ P = (S, R) \,\middle|\, \begin{array}{c} R \subset W_\ell \text{ and} \\ c^{-1}(2m)^{d\ell} \leq \# R \leq c(2m)^{d\ell} \end{array} \right\}.$$

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- $\Gamma_S = \langle s_1 \rangle * \cdots * \langle s_m \rangle$: free group generated by S.
- For $P = (S, R)$ $\Gamma_P = \Gamma_S / \overline{R}$, $\overline{R}$ denotes the normal closure of R.

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Note. W_ℓ contains unreduced words. ($s_i s_i^{-1}$ or $s_j^{-1} s_j$ may appear). $\quad \therefore \#W_\ell = (2m)^\ell$.

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Note. For any finitely presented group Γ, we can find m, ℓ, d so that $\exists P \in P(m, \ell, d)$ s.t. $\Gamma = \Gamma_P$

Thm 1

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Thm 1. Take $\mathbb{N} \ni k > C^6/2$ and $d > \frac{k}{2k+1}$. Then

$$\lim_{\ell\to\infty} \frac{\# \left\{ P \in P(m,\ell,d) \;\middle|\; \begin{array}{c} \forall \mathrm{UL}(C) \text{ action of} \\ \Gamma_P \text{ has a fixed point} \end{array} \right\}}{\# P(m,\ell,d)} = 1.$$

i.e., "any UL(C) action of random groups with density $> k/(2k+1)$ in the plain word model have fixed point."

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(Ollivier '04) $d < 1 - \frac{1}{2}\log_{2m}(8m-4) \nearrow \frac{1}{2}\ (m \to \infty)$

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$\longrightarrow$ Hyperbolicity is very common property.

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(Żuk, Kotowski-Kotowski '13) r. g. in the density model with $d > 1/3$ has (T).
(Nowak, 2015) Let $C < \sqrt{2}$. Any UL(C) action of r. g. in the density model on $\mathcal{H}$ with $d > 1/3$ has a fixed point.

Thm 2.

Thm 2. Fix $C > 0$, $0 \leq \eta < 1/10$.
Any action of random groups in the graph model on $\mathcal{H}$ satisfying

$$\forall \gamma \in \Gamma, \ \mathrm{Lip}(\rho(\gamma)) \leq C l([\gamma])^{\eta}$$

has a fixed point, where

$$l([\gamma]) = \min\{l(\xi\gamma\xi^{-1}) \mid \xi \in \Gamma\},$$

and $l(\gamma')$ denotes the word length of γ'.

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Cor. (Gromov '03) Fix $C \geq 1$. Any UL(C) action of random groups in the graph model on $\mathcal{H}$ has a fixed point.

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For an S-labeling α,
$R_\alpha = \{\alpha(e_1)\ldots\alpha(e_r) \in \Gamma_S \mid (e_1, \ldots, e_r)$ cycle in $G\}$,
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Consider a sequence of graphs $\{G_\ell\}_{\ell\in\mathbb{N}}$ satisfying
(1) $|V_\ell| \to \infty$ $(\ell \to \infty)$,
(2) $\exists d, \forall \ell, \forall u \in V_\ell, 2 \le \deg(u) \le d$,
(3) $\exists \lambda > 0, \forall \ell, \lambda_1(G_\ell,\mathbb{R}) \ge \lambda$ (highly-connected),
(4) $\mathrm{girth}(G_\ell) \to \infty$ $(\ell \to \infty)$,
(5) $\exists D > 0, \mathrm{diam}(G_\ell) \le D\,\mathrm{girth}(G_\ell)$,
(6) a few more conditions for infinite hyperbolicity.

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<u>Thm 2</u>. Fix $C > 0$, $0 \leq \eta < 1/10$. Then

$$\lim_{\ell\to\infty} \frac{\#\left\{ \alpha \in \mathcal{A}_\ell \;\middle|\; \begin{array}{c} \Gamma_\alpha \text{ is infinite hyperbolic, and} \\ \forall \text{ action of } \Gamma_\alpha \text{ on } \mathcal{H} \\ \text{satisfying } (*) \text{ has a fixed point} \end{array} \right\}}{\#\mathcal{A}_\ell} = 1,$$

where

$$\forall \gamma \in \Gamma,\ \mathrm{Lip}(\rho(\gamma)) \leq Cl([\gamma])^\eta \quad (*)$$

About the proof

1st step: If $\rho(\Gamma)$ has no fixed point, we can find an orbit stretched tightly. Precisely: taking a ultralimit of conjugates of ρ, one can find a ρ_∞ admitting an equivariant harmonic map $f: \Gamma \to \mathcal{H}$ i.e.,

$$f(\gamma x) = \rho_\infty(\gamma) f(x) \quad \forall \gamma, x \in \Gamma$$

$$\sum_{s \in S} \frac{1}{\# S} f(xs) = f(x) \quad \forall x \in \Gamma$$

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2nd step: One can expect that if there is a graph G with $\lambda_1(G, \mathbb{R})$ away from 0 in the Cayley graph of Γ, then its orbit cannot be stretched tightly. (such a Γ does not admit a nontrivial equivariant harmonic map.)

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1st step: If $\rho(\Gamma)$ has no fixed point, we can find an orbit stretched tightly.

2nd step: One can expect that if there is a graph G with $\lambda_1(G, \mathbb{R})$ away from 0 in the Cayley graph of Γ, then its orbit cannot be stretched tightly.

3rd step: Prove that under the assumption of Thm's, such a graph G can be found in Γ with overwhelming probability.