Noncommutative topology and quantum groups

Takeshi Katsura

Faculty of Science and Technology, Keio University

27th / June / 2017
Boston University / Keio University
Workshop 2017
Geometry and Mathematical Physics
Boston University 26th-30th / June / 2017

Plan

PART I. Noncommutative topology

§1: C*-algebras and Gelfand transform

§2: Deformation quantization

§3: Group C*-algebras

§4: Crossed products

§5: Warped crossed products

PART II. Quantum groups

PART I. Noncommutative topology

- A suitable algebra A of functions on a space X remembers the space X.
- Transfer a theory of spaces X to a theory of the algebras A of functions on X.
- Extend such theories from commutative algebras to noncommutative algebras.
- Regard a noncommutative algebra as an algebra of “functions” on a “noncommutative space”.

§1: Definition of C*-algebras

Definition

*-algebra $:=$ algebra A over complex numbers $\mathbb{C}$
w/ involution $A \ni x \mapsto x^* \in A$

$$(\alpha x)^* = \overline{\alpha} x^* \quad (x+y)^* = x^* + y^*$$
$$(xy)^* = y^* x^* \quad (x^*)^* = x$$

Definition

C*-algebra $:= *$-algebra A w/ complete norm $\|\cdot\|$
satisfying $\|xy\| \leq \|x\|\|y\|$, $\|x^*x\| = \|x\|^2$

In this talk, every C*-algebra is unital
i.e. has a multiplicative identity 1

§1: Examples of C*-algebras

Example (C*-algebra)

- complex numbers $\mathbb{C}$
- the matrix algebras $M_n(\mathbb{C})$ ($n = 1, 2, \ldots$)
- the algebra $B(\mathcal{H})$ of bounded operators on a Hilbert space $\mathcal{H}$

Theorem (Gelfand-Naimark)

Every C-algebra is isomorphic to a C*-subalgebra of $B(\mathcal{H})$.*

§1: Gelfand transform

Example (commutative C*-algebra)

X: compact space

$C(X) := \{f : X \to \mathbb{C} \mid \text{continuous}\}$

$\mathbb{T} := \{z \in \mathbb{C} \mid |z| = 1\}$ n-torus $\mathbb{T}^n$

n-sphere S^n real projective n-space $\mathbb{R}P^n$

Theorem (Gelfand)

Every unital commutative C-algebra is isomorphic to $C(X)$ for a unique compact space X.*

§2: Deformation quantization

"continuous" map $[0,\infty) \ni q \mapsto A_q$: C*-algebra
s.t. $A_0 \cong C(X)$: commutative C*-algebra

Recipe for deformation quantization
of a compact space X:

(i) $C(X) \cong C^*(f_1,\ldots,f_n \mid \mathcal{R}_{\mathrm{c}} \cup \mathcal{R}_{\mathrm{o}})$
$\mathcal{R}_{\mathrm{c}}$: commutation relations $f_k f_l = f_l f_k$,
$f_k^* f_l = f_l f_k^*$
$\mathcal{R}_{\mathrm{o}}$: other relations among $\{f_k\}$

(ii) $A_q := C^*(a_1,\ldots,a_n \mid \mathcal{R}_c^q \cup \mathcal{R}_o)$
$\mathcal{R}_{\mathrm{c}}^q$: relations $a_k a_l = e^{iq} a_l a_k$ etc.

§2: Generators and relations

X: compact space $\qquad f_1, \ldots, f_n \in C(X)$

$f_1, \ldots, f_n$ genarates $C(X)$

$$\iff \boldsymbol{f} := (f_1, \ldots, f_n) \colon X \to \mathbb{C}^n \text{ is injective}$$

other relations $\mathcal{R}_0$ among $\{f_k\}$

$$\leftrightsquigarrow \text{relations determining } \boldsymbol{f}(X) \subset \mathbb{C}^n$$

Example

- $C(\mathbb{T}) \cong C^*(u \mid u^*u = uu^* = 1) \qquad u$: unitary
- $C(\mathbb{T}^n) \cong C^*(u_1, \ldots, u_n \mid u_k u_l = u_l u_k$
  u_k: unitary$)$
- $C(S^{n-1}) \cong C^*(h_1, \ldots, h_n \mid h_k h_l = h_l h_k$
  $h_k = h_k^*, \ \sum_{k=1}^n h_k^2 = 1)$

§2: Examples of deformation quantization

- Noncommutative 2-torus

  $\mathbb{R} \ni \theta \mapsto A_\theta \qquad (A_0 = C(\mathbb{T}^2))$

  $A_\theta := C^*(u, v \mid uv = e^{2\pi i\theta} vu,\ \ u, v\text{: unitary})$

  ♠ $K_n(A_\theta) = K_n(A_0) = K^{-n}(\mathbb{T}^2) = \mathbb{Z}^2$ $(n = 0, 1)$

- Noncommutative 3-sphere

  $\mathbb{R} \ni \theta \mapsto B_\theta \qquad (B_0 = C(S^3))$

  $C(S^3) \cong \qquad B_\theta := C^*(u, v \mid uv =$
  $e^{2\pi i\theta} vu,\ u^*u = uu^*$
  $v^*v = vv^*,\ u^*u + v^*v = 1)$

  ♠ $U(1)$-bundle over S^2, Heegaard splitting, noncommutative lens space

§3: Group C*-algebras

Γ: (discrete) group

$C^*(\Gamma) := C^*(\{u_t\}_{t\in\Gamma} \mid u_t\text{: unitary, } u_t u_s = u_{ts})$

$C^*(\Gamma)$ remembers all unitary representations of Γ

$C^*(\Gamma)$ is commutative $\iff$ Γ is abelian

in this case $C^*(\Gamma) \cong C(\widehat{\Gamma})$

where $\widehat{\Gamma}$: the Pontryagin dual of Γ

Example

$C^*(\mathbb{Z}) \cong C(\mathbb{T})$, $\quad C^*(\mathbb{Z}^n) \cong C(\mathbb{T}^n)$

F: finite abelian $\Rightarrow C^*(F) \cong C(\widehat{F}) \cong C(F) \cong \mathbb{C}^{|F|}$

$C^*(\mathfrak{S}_3) \cong \mathbb{C} \oplus \mathbb{C} \oplus M_2(\mathbb{C})$

§3: Twisted Group C*-algebra

$\omega\colon \Gamma \times \Gamma \to \mathbb{T}$: 2-cocycle of Γ

$$(\omega(t_0, t_1)\omega(t_0, t_1t_2)^{-1}\omega(t_0t_1, t_2)\omega(t_1, t_2)^{-1} = 1)$$

$$C^*_\omega(\Gamma) := C^*(\{u_t\}_{t\in\Gamma} \mid u_t\text{: unitary, } u_tu_s = \omega(t, s)u_{ts})$$

Example

$\theta \in \mathbb{R}$, $\quad \omega_\theta\colon \mathbb{Z}^2 \times \mathbb{Z}^2 \to \mathbb{T}$ by

$$\omega_\theta((n, m), (k, l)) = e^{2\pi i m k \theta}$$

$$C^*_{\omega_\theta}(\mathbb{Z}^2) = A_\theta\text{: noncommutative 2-torus}$$

In part II, we discuss examples for
the Klein 4-group $K \cong (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$.

§4: Crossed products

Γ: (discrete) group,
X: compact space,
$\alpha: \Gamma \curvearrowright X$: action
(i.e. $\Gamma \ni t \mapsto \alpha_t \in \mathrm{Homeo}(X)$: group hom)
$\rightsquigarrow C(X) \rtimes_\alpha \Gamma$: crossed product
C*-algebra generated
by $C(X)$ and unitaries $\{u_t\}_{t \in \Gamma}$
satisfying $u_t u_s = u_{ts}$ and $f u_t = u_t (f \circ \alpha_t)$

If α: trivial ($\alpha_t = \mathrm{id}$) and Γ is abelian,
$C(X) \rtimes_\alpha \Gamma \cong C(X) \otimes C^*(\Gamma) \cong C(X \times \widehat{\Gamma})$

§4: noncommutative 2-torus revisited

$C(X) \rtimes_\alpha \Gamma$ "$\cong$" $C(X/\Gamma) \otimes M_{|\Gamma|}(\mathbb{C})$

$\theta \in \mathbb{R} \rightsquigarrow \alpha_\theta \colon \mathbb{Z} \curvearrowright \mathbb{T}$: action
by $\mathbb{Z} \ni n \mapsto (z \mapsto e^{2\pi i n\theta} z) \in \mathrm{Homeo}(\mathbb{T})$

Then, $C(\mathbb{T}) \rtimes_{\alpha_\theta} \mathbb{Z} \cong A_\theta$: noncommutative 2-torus

In this way, one can get many
deformation quantization of spaces like $X \times \mathbb{T}$.

Noncommutative crossed product,
twisted crossed product, . . .

§5: Warped crossed products

X: compact space,
$\alpha\colon \mathbb{Z} \curvearrowright X$: action $\leftrightsquigarrow$ $\alpha_1 \in \mathrm{Homeo}(X)$
$\mathcal{L}$: Hermitian line bundle over X
$\rightsquigarrow C(X) \rtimes_{\alpha,\mathcal{L}} \mathbb{Z}$: warped crossed product
(Søren Eilers-TK)

α: trivial ($\alpha_1 = \mathrm{id}$), $\Rightarrow$ $C(X) \rtimes_{\alpha,\mathcal{L}} \mathbb{Z} \cong C(\mathcal{L}^1)$
where $\mathcal{L}^1 := \{v \in \mathcal{L} \mid \|v\| = 1\}$
principal $U(1)$-bundle associated with $\mathcal{L}$

$X = S^2$ and $\mathcal{L}$ varies (e.g. Hopf fibration),
$\rightsquigarrow$ deformation quantization of S^3 or
the Lens spaces (Matsumoto-Tomiyama)

PART II. Quantum groups

PART I:

X: compact space $\leftrightsquigarrow$ $C(X)$: C*-algebra

PART II

G: compact group

i.e. G: compact space $\leftrightsquigarrow$ $C(G)$: C*-algebra

with product $G \times G \to G$ with coproduct

$\Delta \colon C(G) \to C(G \times G)$

by $\Delta(f)(t,s) = f(ts)$

$\spadesuit\, C(G \times G) \cong C(G) \otimes C(G)$

(c.f. Hopf algebra)

§1. Definition and Examples

Definition

compact quantum group := C*-algebra A with coproduct $\Delta\colon A \to A \otimes A$ s.t.

$$(\Delta \otimes \mathrm{id}_A) \circ \Delta = (\mathrm{id}_A \otimes \Delta) \circ \Delta \text{ and “}\exists \text{ inverse”}$$

Example (compact quantum group)

- $C(G)$ for a compact group G
- $C^*(\Gamma)$ for a discrete group Γ with $\Delta(u_t) = u_t \otimes u_t$
- $SU_q(n), \ldots$

§1: Duality Theorem

♠ Dual quantum group

a compact quantum group (A, Δ)

$\rightsquigarrow$ a discrete quantum group $(\widehat{A}, \widehat{\Delta})$,

a discrete quantum group (A, Δ)

$\rightsquigarrow$ a compact quantum group $(\widehat{A}, \widehat{\Delta})$,

Example

F: finite group

$(C(F), \Delta)$ and $(C^*(F), \Delta)$ are

the duals of each others.

§2: Definition of free symmetric groups

$$\mathfrak{S}_n \hookrightarrow U(n) \hookrightarrow M_n(\mathbb{C}) \cong \mathbb{C}^{n^2}$$

$$C(\mathfrak{S}_n) \cong C^*(\{p_{k,l}\}_{k,l=1}^{n} \mid \mathcal{R}_{\mathrm{c}} \cup \mathcal{R}_{\mathrm{o}})$$

$\mathcal{R}_{\mathrm{c}}$: commutation relations $p_{k,l}p_{k',l'} = p_{k',l'}p_{k,l}$

$\mathcal{R}_{\mathrm{o}}$: $p_{k,l}$ projection (i.e. $p_{k,l}^* = p_{k,l}^2 = p_{k,l}$)

$$\sum_{l=1}^{n} p_{k,l} = 1 \ (\forall k), \qquad \sum_{k=1}^{n} p_{k,l} = 1 \ (\forall l)$$

$$\Delta(p_{k,l}) = \sum_{m=1}^{n} p_{k,m} \otimes p_{m,l}$$

Definition (Wang '98)

$(A_s(n), \Delta)$: free symmetric group

$$A_s(n) := C^*(\{p_{k,l}\}_{k,l=1}^{n} \mid \mathcal{R}_{\mathrm{o}})$$

magic square C*-algebra

§2: Results on free symmetric groups

Proposition

$n = 1,2,3 \;\Rightarrow\; (A_s(n),\Delta) \cong (C(\mathfrak{S}_n),\Delta)$

$n \geq 4 \;\Rightarrow\; A_s(n)$: *noncommutative,* ∞*-dim.*

$n \geq 5 \;\Rightarrow\; (A_s(n),\Delta)$: *not coamenable,*

$A_s(n)$: *not exact*

Proposition (Banica-Collins, Ogawa)

$$A_s(4) \hookrightarrow C(\mathbb{R}P^3) \otimes M_4(\mathbb{C}) \cong C(\mathbb{R}P^3, M_4(\mathbb{C}))$$

§3: Magic square C*-algebra of size 4

$$A_s(4) = C^*(\{p_{k,l}\}_{k,l=1}^{4} \mid \mathcal{R}_o)$$

$\mathcal{R}_o$: $p_{k,l}$ projection (i.e. $p_{k,l}^* = p_{k,l}^2 = p_{k,l}$)

$\sum_{l=1}^{4} p_{k,l} = 1$ $(\forall k)$, $\sum_{k=1}^{4} p_{k,l} = 1$ $(\forall l)$

Definition

$K := \{t_1, t_2, t_3, t_4\} \subset \mathfrak{S}_4$: Klein 4-group

$t_2 = (12)(34)$, $t_3 = (13)(24)$, $t_4 = (14)(23)$

$\alpha \colon \mathfrak{S}_4 \times \mathfrak{S}_4 \curvearrowright A_s(4)$: action

$$\alpha_{\sigma,\tau}(p_{k,l}) = p_{\sigma(k),\tau(l)}$$

Restrict α to $K \times K \subset \mathfrak{S}_4 \times \mathfrak{S}_4$.

§3: 2-cocycle of K

$\varepsilon_{k,l} :=$

$k \diagdown l$	1	2	3	4
1	1	1	1	1
2	1	−1	1	−1
3	1	−1	−1	1
4	1	1	−1	−1

$\omega\colon K \times K \to \mathbb{T}$: 2-cocycle by $\omega(t_k, t_l) := \varepsilon_{k,l}$

$C^*_\omega(K) \cong M_2(\mathbb{C})$ (c.f. $C^*(K) \cong \mathbb{C}^4$)

$c_1, c_2, c_3, c_4 \in M_2(\mathbb{C})$: unitary

$$t_k t_l = t_m \rightsquigarrow c_k c_l = \varepsilon_{k,l} c_m$$

§3: Pauli matrices

Pauli matrix $c_1, c_2, c_3, c_4 \in M_2(\mathbb{C})$

$$c_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad c_2 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},$$

$$c_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad c_4 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

$\{c_1, c_2, c_3, c_4\}$: orthonormal basis of the Hilbert space $(M_2(\mathbb{C}), \mathrm{trace}) \cong (\mathbb{C}^4, \langle\cdot|\cdot\rangle)$

$S^3 \ni (a_1, a_2, a_3, a_4) \mapsto \sum_{k=1}^4 a_k c_k \in SU(2)$

$SU(2)/\{\pm 1\} \cong \mathbb{R}P^3$

§3: Real projective space

$S^{n-1} \hookrightarrow \mathbb{R}^n \rightsquigarrow h_1, \ldots, h_n \in C(S^{n-1})$

$h_k h_l$ factors through $\mathbb{R}P^{n-1} \rightsquigarrow f_{k,l} \in C(\mathbb{R}P^{n-1})$

Proposition

$$C(\mathbb{R}P^{n-1}) \cong C^*(\{f_{k,l}\}_{k,l=1}^{n} \mid \mathcal{R})$$

$$\mathcal{R}\colon f_{k,l} = f_{k,l}^* = f_{l,k}, \qquad f_{k,l}f_{k',l'} = f_{k,k'}f_{l,l'},$$

$$\textstyle\sum_{k=1}^{n} f_{k,k} = 1$$

Compare it with

Proposition

$$M_n(\mathbb{C}) = C^*(\{e_{k,l}\}_{k,l=1}^{n} \mid \mathcal{R}')$$

$$\mathcal{R}'\colon e_{k,l}^* = e_{l,k} \qquad e_{k,l}e_{k',l'} = \delta_{l,k'}e_{k,l'}$$

$$\textstyle\sum_{k=1}^{n} e_{k,k} = 1$$

§3: Main Theorem

Theorem (Ogawa)

$$A_s(4) \rtimes_{\alpha,\omega\times\omega} (K \times K) \cong C(\mathbb{R}P^3) \otimes M_4(\mathbb{C})$$

generators: $\{p_{k,l}\} \cup \{u_{k,l}\}$ v.s. $\{f_{k,l}\} \cup \{e_{k,l}\}$

Dual action of $\widehat{K \times K} \cong K \times K$ yields

Corollary

$$A_s(4) \cong C(\mathbb{R}P^3, M_4(\mathbb{C}))^{K\times K}$$