The Rasmussen invariant of Torus knots

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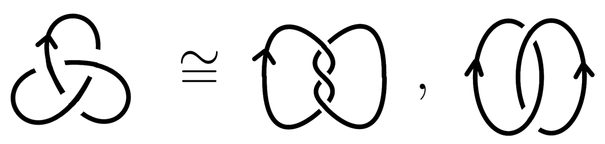
Introduction

- Rasmussen invariant gives proof of Milnor conjecture.
- The idea of this invariant comes from Khovanov cohomology.
- I'll introduce definition of Rasmussen invariant and proof of Milnor conjecture.

Definition (Knot, Link, Equivalence)

- Knot K is an oriented simple closed curve in S^3.
- Link L is a collection of knots.
- $L_1, L_2 \subset S^3$:Link, if there is a continuous transform from L_1 to L_2, we say they are equivalent ($L_1 \cong L_2$).
- Link diagram is a projection of link like pictures below.



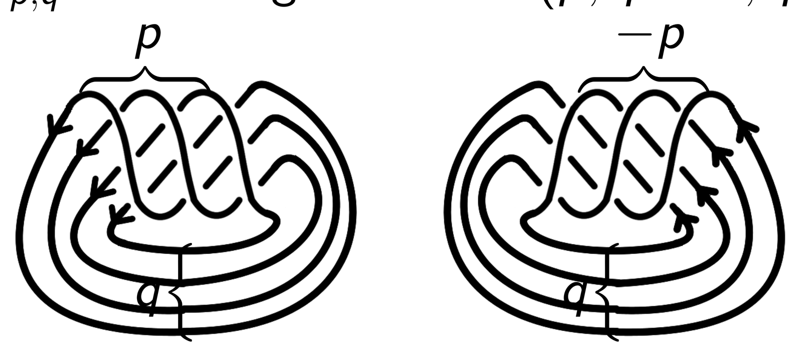
- Equivalence of links corresponds to some elementary moves of diagrams (called Reidemeister moves).
- So, we only think about link diagram.

Milnor Conjecture, [1]

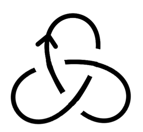
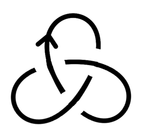
The genus of torus knot $g(T_{p,q})$ $(p, q \in \mathbb{Z}, q > 0)$ is $(|p| - 1)(q - 1)/2$

Torus knot

- (p, q)-Torus link $T_{p,q}$ has a diagram below $(p, q \in \mathbb{Z}, q > 0)$.



$p > 0$ $p < 0$

- Torus link $T_{p,q}$ becomes knot if and only if $\gcd(p, q) = 1$.
-  is $(-2, 3)$-torus knot. So, I'll explain using  for example.

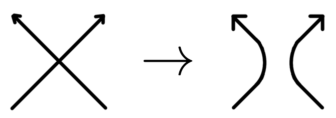
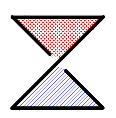
Definition (genus of knot)

The genus g of knot K is defined as

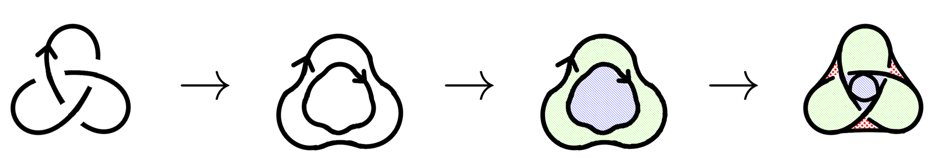
$$g(K) = \min\{g(F) | \text{connected oriented surface } F \subset S^3, \partial F = K\}$$

Seifert algorithm, [2]

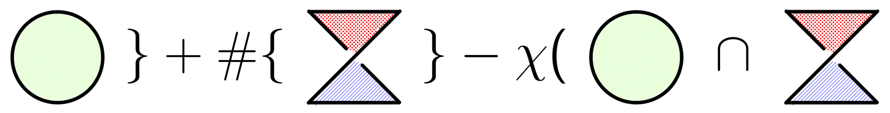
Seifert algorithm gives connected oriented surface F which ∂F is K.

1. Do oriented splice  on all crossings of diagram.
2. Consider discs bounded by the circles appeared in step 1.
3. Connect these discs by twist band  corresponding to crossings of K.

This is the surface F which we want.



Calculating genus of F is easy from the algorithm.



$$2g(F) = 1 - \chi(F) = 1 - (\#\{\bigcirc\} + \#\{\bowtie\} - \chi(\bigcirc \cap \bowtie))$$
$$= 1 + n(K) - k$$

(k = the number of circle appeared in step 1, $n(K)$ = the number of crossing of K). By definition,

$$g(K) \le g(F)$$
$$2g(T_{-2,3}) \le 2 = (2 - 1)(3 - 1)$$

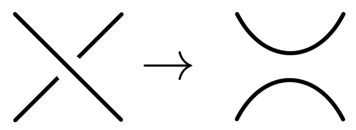
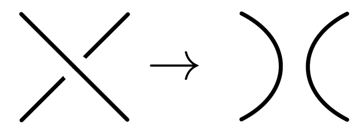
So, to prove Milnor conjecture, we need

$$2g(T_{p,q}) \ge (|p| - 1)(q - 1)$$

To prove it, we use Rasmussen invariant. Rasmussen invariant is defined as grade of Lee cohomology.

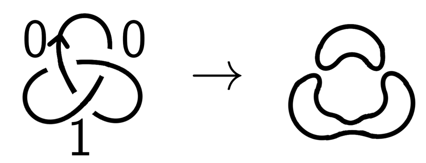
Lee cohomology, [3]

Let $n_+(K) := \#\left\{ \nwarrow\!\!\!\!\nearrow \in K \right\}$, $n_-(K) := \#\left\{ \nwarrow\!\!\!\!\nearrow \in K \right\}$, $n(K) = n_+(K) + n_-(K)$ = the number of crossings of K.

1. Do 0-splice  or 1-splice  on each crossing.

We get $2^{n(K)}$ diagrams which not have crossings.

2. Associate a module $A^{\otimes c}\{-l + 2n_-(K) - n_+(K)\}$ to the c circle diagram which is made by l times 1-splice.



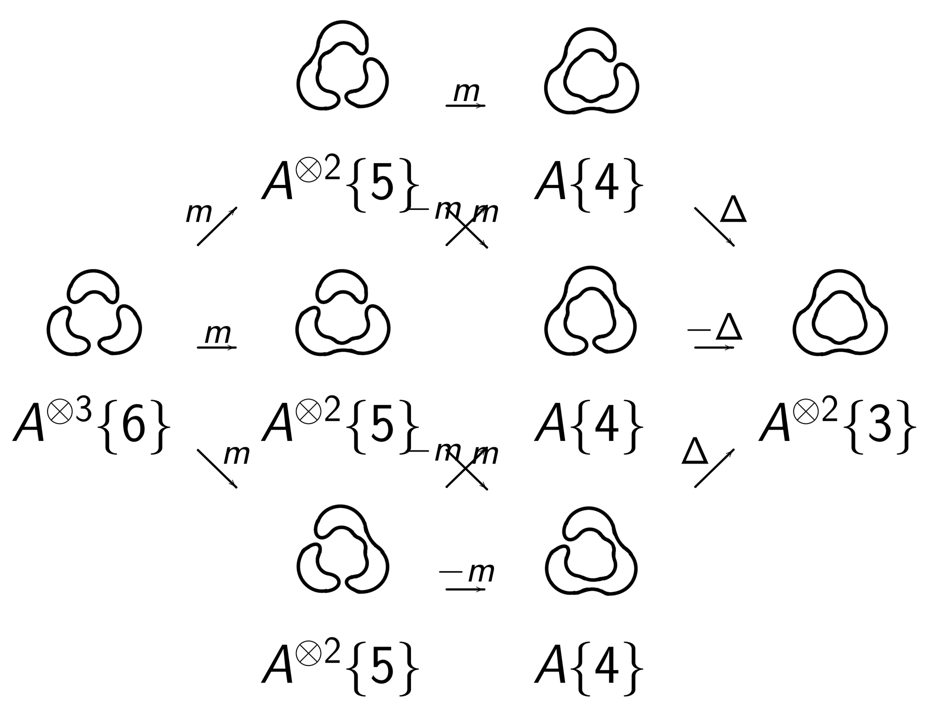
$\to A^{\otimes 2}\{-1 + 6\}$

- $A := \langle \mathbf{1}, X\rangle$: graded $\mathbb{Q}$-module. $(\deg(\mathbf{1}) = 1, \deg(X) = -1)$
- $\{k\}$ means degree $-k$, like $\deg(\mathbf{1} \in A\{k\}) = 1 - k$.

3. Consider two diagrams which are able to achieve by change resolution of a crossing from 0-splice to 1-splice. Connect between corresponding modules by the map $m : A^{\otimes 2} \to A$ or $\Delta : A \to A^{\otimes 2}$.

$$m(\mathbf{1} \otimes \mathbf{1}) = m(X \otimes X) = \mathbf{1} \qquad \Delta(\mathbf{1}) = \mathbf{1} \otimes X + X \otimes \mathbf{1}$$
$$m(\mathbf{1} \otimes X) = m(X \otimes \mathbf{1}) = X \qquad \Delta(X) = X \otimes X + \mathbf{1} \otimes \mathbf{1}$$

Define them as differential d. Some change of sign is necessary to make $d \circ d = 0$



4. Take direct sum of each column of $\mathbb{Q}$-modules appeared in this table. We get sequence of $\mathbb{Q}$-modules, let it $C(K)$.

$$C(T_{-2,3}) = (0 \to A^{\otimes 3}\{6\} \xrightarrow{d^0} \bigoplus_3 A^{\otimes 2}\{5\} \xrightarrow{d^1} \bigoplus_3 A\{4\} \xrightarrow{d^2} A^{\otimes 2}\{3\} \xrightarrow{d^3} 0)$$

5. Its cohomology group $H^*(C(K))$ is Lee cohomology.

Rasmussen invariant, [4]

Let grade s,

$$s([x]) := \min\{j \in \mathbb{Z} | \exists[y] = [x] \text{ s.t. } y \in \langle b \in C(K) | \deg(b) \ge j\rangle\}$$
$$s_{\max}(K) := \max\{s([x]) \in \mathbb{Z} | [x] \in H^*(C(K))\}$$
$$s_{\min}(K) := \min\{s([x]) \in \mathbb{Z} | [x] \in H^*(C(K))\}$$

These are link invariants. If K is a knot,

$$s_{\max}(K) = s_{\min}(K) + 2$$

So, we can define Rasmussen invariant $s(K)$.

Definition (Rasmussen, [4])

K:Knot,

$$s(K) := s_{\min}(K) + 1 = s_{\max}(K) - 1$$

Consider the table of $C(T_{-2,3})$. Its j-row is degree of each bases.

$C(T_{-2,3})$	0	1	2	3
-1				$\mathbf{1} \otimes \mathbf{1}$
-3	$\mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1}$	$\mathbf{1} \otimes \mathbf{1}$	$\mathbf{1}$	$\mathbf{1} \otimes X$ $X \otimes \mathbf{1}$
-5	$\vdots$	$\vdots$	X	$X \otimes X$

From this table, we can see $\mathbf{1} \otimes \mathbf{1} \in A^{\otimes 2}\{3\}$ gives maximal degree of $C(T_{-2,3})$. d^3 is a 0-map and $[\mathbf{1} \otimes \mathbf{1}] \in H^*(C(T_{-2,3}))$. So, we can see

$$s_{\max}(T_{-2,3}) = \deg(\mathbf{1} \otimes \mathbf{1}) = 2 - 3 = -1$$

Then,

$$s_{\min}(T_{-2,3}) = -3, \; s(T_{-2,3}) = -2 = -(2 - 1)(3 - 1)$$

Theorem (Rasmussen, [4])

Let slice genus
$g_*(K) := \min\{g(F) | \text{connected oriented surface } F \subset B^4, \; \partial F = K\}$*, then,*

$$|s(K)| \le 2g_*(K)$$

By definition of slice genus,

$$g_*(K) \le g(K)$$

So,

$$(2 - 1)(3 - 1) = |s(T_{-2,3})| \le g(T_{-2,3})$$

In general, we can see $s(T_{-p',q}) = -(p' - 1)(q - 1)$ $(p' > 0)$ because $\mathbf{1} \otimes \cdots \otimes \mathbf{1} \in A^{\otimes q}\{-p'(q - 1)\}$ gives maximal degree of $H^*(C(T_{-p',q}))$. So, when $p < 0$ Milnor Conjecture is proved.

[1] J. Milnor, Singular points of complex hypersurfaces, Princeton University Press, (1969)

[2] H. Seifert, Über das Geschlet von Knoten, Math. Ann., 110, 571-592, (1934)

[3] E. S. Lee, An endomorphism of the Khovanov invariant, Advances in mathematics, Vol. 197, Issue 2, 554-586, (2005)

[4] J. Rasmussen, Khovanov homology and the slice genus, Invent. math., 182, 419-447, (2010)