On the relation between amenability of discrete groups and nuclearity of group C*-algebras

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Definition (C*-algebra)

A Banach space A is a C*-algebra.
:⇔ A is a $\mathbb{C}$-algebra with an involution $A \ni x \mapsto x^* \in A$ such that

- $\|xy\| \leq \|x\|\|y\|$,
- $\|x^*x\| = \|x\|^2$ (C*-condition),
- $(\alpha x + \beta y)^* = \bar{\alpha}x^* + \bar{\beta}y^*$,
- $(xy)^* = y^*x^*$, $(x^*)^* = x$

for all $\alpha, \beta \in \mathbb{C}$, $x, y \in A$.
We always assume that a C-algebra A has a unit (i.e. multiplicative identity)* $1_A \in A$.

Example (C*-algebra)

- X: compact Hausdorff space.
  $C(X) := \{f : X \to \mathbb{C} \mid f \text{ is continuous}\}$
- H: Hilbert space over $\mathbb{C}$.
  $L(H) := \{x : H \to H \mid x \text{ is a bounded linear}\}$
- (Matrix algebra) A: C*-algebra.
  $M_n(A) := \left\{ [a_{i,j}]_{i,j=1}^n \;\middle|\; a_{i,j} \in A \right\}$

Definition (Positivity)

A: C*-algebra, $x \in A$.
$x \geq 0$:⇔ $x = y^*y$ for some $y \in A$.

Definition (Completely positive map)

A linear map $\varphi : A \to B$ is *unital*.
:⇔ $\varphi(1_A) = 1_B$.
A linear map $\varphi : A \to B$ is *positive*.
:⇔ $x \geq 0$ implies $\varphi(x) \geq 0$ for all $x \in A$.
φ is *completely positive* (c.p.).
:⇔ $M_n(A) \ni [a_{i,j}] \to [\varphi(a_{i,j})] \in M_n(B)$ is positive for every $n \in \mathbb{N}$.

Definition (Nuclearity)

A separable C*-algebra A is nuclear.
:⇔ $\exists \varphi_i : A \to M_{n_i}(\mathbb{C}), \psi_i : M_{n_i}(\mathbb{C}) \to A$ sequences of unital c.p. maps such that

$$\lim_{i\to\infty} \|x - \psi_i(\varphi_i(x))\| = 0 \qquad (\forall x \in A).$$

Example (Nuclear C*-algebra)

- Finite dimensional C*-algebra $M_n(\mathbb{C})$.
- Commutative C*-algebra $C(X)$.
- Direct sums and inductive limits of nuclear C*-algebras.

Definition (Left regular representation)

Γ: discrete group. $s \in \Gamma$.
$\lambda_s \in B(\ell^2\Gamma)$ is defined by $\lambda_s(f)(t) := f(s^{-1}t)$, where $f \in \ell^2\Gamma, t \in \Gamma$.
$\lambda : \Gamma \ni s \mapsto \lambda_s \in B(\ell^2\Gamma)$ is called *left regular representation*.

Definition (Reduced group C*-algebra)

Γ: discrete group.

$$C^*_\lambda(\Gamma) := \overline{\left\{ \textstyle\sum_{s\in\Gamma, \text{finite sum}} a_s\lambda_s \in B(\ell^2\Gamma) \mid a_s \in \mathbb{C} \right\}}^{\|\cdot\|}.$$

$C^*_\lambda(\Gamma)$ is a C*-algebra generated by $\{\lambda_s\}_{s\in\Gamma}$.

Definition (Amenabiliy)

Γ is amenable.
:⇔ $\exists \mu : \ell^\infty\Gamma \to \mathbb{C}$ linear map s.t.

1. $\mu(1) = 1$.
2. $\forall f \in \ell^\infty\Gamma,\ f \geq 0 \Rightarrow \mu(f) \geq 0$.
3. $\forall s \in \Gamma, \forall f \in \ell^\infty\Gamma,\ \mu(sf) = \mu(f)$.

where $sf(t) := f(s^{-1}t)$

μ is called an invariant mean of Γ.

Theorem ([1, Theorem 2.6.8.])

The followings are equivalent;

1. Γ is amenable.
2. $C^*_\lambda(\Gamma)$ is nuclear.

Example (Discrete Abelian Group)

Every discrete Abelian group Γ is amenable.

Proof. If Γ is an Abelian group, then $C^*_\lambda(\Gamma)$ is commutative. Since every commutative C*-algebra is nuclear, Γ is amenable by the previous theorem. □

Example (Free group)

Let F_2 be a free group generated by two elements $\{a, b\}$. Then, F_2 is nonamenable. Therefore, $C^*_\lambda(F_2)$ is nonnuclear.

Proof. Let $A^+, A^-, B^+, B^- \subset F_2$ be a set of reduced words which start with a, a^{-1}, b, b^{-1} respectively. For $C := \{1, b, b^2, \dots\}$,

$$\begin{aligned} F_2 &= A^+ \sqcup A^- \sqcup \left(B^+ \setminus C\right) \sqcup \left(B^- \cup C\right) \\ &= A^+ \sqcup aA^- = b^{-1}\left(B^+ \setminus C\right) \sqcup \left(B^- \cup C\right). \end{aligned}$$

If $\mu : \ell^\infty F_2 \to \mathbb{C}$ is an invariant mean,

$$\begin{aligned} 1 = \mu(1) &= \mu(\chi_{A^+}) + \mu(\chi_{A^-}) + \mu(\chi_{B^+\setminus C}) + \mu(\chi_{B^-\cup C}) \\ &= \mu(\chi_{A^+}) + \mu(a\chi_{A^-}) + \mu(b^{-1}\chi_{(B^+\setminus C)}) + \mu(\chi_{B^-\cup C}) \\ &= \mu(\chi_{A^+} + \chi_{aA^-}) + \mu(\chi_{b^{-1}(B^+\setminus C)} + \chi_{B^-\cup C}) \\ &= \mu(1) + \mu(1) = 2. \end{aligned}$$

Therefore, F_2 is not amenable. Nonnuclearity of $C^*_\lambda(F_2)$ follows from the previous theorem. □

Reference

[1] N.P. Brown and N. Ozawa. *C*-algebras and Finite-dimensional Approximations.* Graduate studies in mathematics. American Mathematical Soc., 2008.