Dualizing the relative higher index and almost flat bundles

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Outlines

Introduction

Homotopy invariance of the rational Pontrjagin class

$\varphi\colon M \to N$: homotopy equivalence of closed manifolds.

Q. $\varphi^* p_k(TN) = p_k(TM) \in H^*(M;\mathbb{Q})$?

A. No.

e.g.) $S(E) \sim S^4 \times S^j$ (E is a $\mathbb{R}$-vector bundle on S^4)

Theorem (Hirzebruch Signature theorem)

$$\langle L(M), [M] \rangle = \mathrm{Sign}(M)$$

In particular it is homotopy invariant.

Let $f\colon M \to B\pi_1(M)$.

Conjecture (Noivikov conjecture)

$f_*(L(M) \cap [M]) \in H_*(B\pi_1(M);\mathbb{Q})$ *is homotopy invariant.*

Existence of a metric with positive scalar curvature (psc)

Theorem (Kazdan–Warner)

One of the following holds.

1. $\forall s \in C^\infty(M)$ $\exists g$ *a metric s.t.* $\mathrm{Scal}_g = s$.
2. $\exists g$ *a metric s.t.* $\mathrm{Scal}_g = s$ *iff* $\exists x \in M$ *s.t.* $s(x) < 0$ *or* $s \equiv 0$.
3. $\exists g$ *fa metric s.t.* $\mathrm{Scal}_g = s$ *iff* $\exists x \in M$ *s.t.* $s(x) < 0$.

Let M be a closed spin manifold.

Theorem (Atiyah–Singer index theorem)

If M admits a psc metric, then $\langle \hat{A}(M), [M] \rangle = 0$.

Conjecture (Gromov–Lawson conjecture)

If M admits a psc metric, then $f_*(\hat{A}(M) \cap [M]) = 0$.

More precisely, one should use the Real K-theory instead of cohomology.
(Another question: is it necessary and sufficient? (GLR conjecture))

Almost flat vector bundles

Almost flat element of the K^0-group

(M, g): closed Riemannian manifold

Definition

An *ε-flat vector bundle* on M is a hermitian vector bundles E on M such that E admits a connection ∇ with the curvature satisfies

$$\|R^\nabla\| := \sup_{x \in M} \sup_{X, Y \in T_x M} \frac{\|R^\nabla(X, Y)\|_{\mathrm{End}(E_x)}}{\|X\| \|Y\|} < \varepsilon.$$

For $\alpha \in \mathrm{K}^0(M)$,

$$\text{K-area}(M, g, \alpha) := \sup_{\alpha = [E^+] - [E^-]} \min \left\{ \frac{1}{\|R^{\nabla^+}\|}, \frac{1}{\|R^{\nabla^-}\|} \right\}.$$

α is almost flat iff $\text{K-area}(M, g, \alpha) = \infty$.

Almost flatness and higher genus

Theorem (Connes–Gromov–Moscovici'91)

$x \in \mathrm{K}^0(M)$: *almost flat. Then,*

$$\mathrm{Sign}_x(M) = \langle \mathrm{ch}(x) L(M), [M] \rangle$$

is homotopy invariant.

More precisely, for $f \colon N \to M$: homotopy equivalence

$$\langle f^* \mathrm{ch}(x) L(N), [N] \rangle = \langle \mathrm{ch}(x) L(M), [M] \rangle.$$

Theorem

$x = [E] - [\mathbb{C}^n_M] \in \mathrm{K}^0(M)$: *almost flat and M admits a psc metric. Then,*

$$0 = \mathrm{Index}(D_E) = \langle \mathrm{ch}(E) p\hat{A}(M), [M] \rangle$$

Q. Does $\mathrm{Im}(f^* \colon \mathrm{K}^*(B\pi_1(M)) \to \mathrm{K}^*(M))$ coincide with the subgroup consisting of all almost flat elements?

Enlargeability

Definition (Gromov–Lawson)

M^n is *enlargeable* if for any $\varepsilon > 0$ there is a finite cover $\overline{M}$ and a ϵ-contracting map $f \colon \overline{M} \to S^n$ of non-zero degree.

- A typical example is $\mathbb{T}^n$.
- Any manifold admitting a metric with non-positive sectional curvature is enlargeable (Gromov–Lawson).

E: vector bundle on S^{2n} corresponding to the Bott generator
Then,

$$\pi_!(f^*E) := \bigsqcup_{x \in M} \bigoplus_{\pi(\overline{x})=x} E_{f(\overline{x})}$$

is a ε-flat vector bundle on M.

Corollary

If a spin manifold M is enlargeable, then M does not admit a psc metric.

Almost flat bundles vs Quasi-representations

Recall the one-to-one correspondence

$$\{\text{flat (herm) vector bundle on } M\} \leftrightarrow \{(\text{unitary}) \text{ representation of } \pi_1(M)\}$$

$$(E, \nabla) \mapsto \text{monodoromy representation},$$

$$\tilde{M} \times_\pi V \leftarrow\!\shortmid (\pi, V).$$

Definition

Γ: discrete group.

$\mathcal{G} \subset \Gamma$: finite subset

$\pi \colon \mathcal{G} \to \mathcal{U}(\mathcal{H})$ is a $(\mathcal{G}, \varepsilon)$-quasirepresentation if

$$\|\pi(s)\pi(t) - \pi(st)\| < \varepsilon$$

for any $s, t, st \in \mathcal{G}$.

Roughly, quasi-representations of $\pi_1(M)$ one-to-one correspond to almost flat bundles on M.

(Mishchenko–Maruilov'95, Dadarlat'14, Carrion–Dadarlat'15)

Dual of the Baum–Connes assembly map

KK-theory

There is an additive category $\mathfrak{KK}$ such that

- $\mathrm{Obj}(\mathfrak{KK}) =$ (separable C*-algebras),
- the morphism set (denoted by $\mathrm{KK}(A, B)$) is

$$\mathrm{KK}(\mathbb{C}, A) \cong \mathrm{K}_0(A) = G\{\text{fin. gen. proj. } A\text{-module }\},$$
$$\mathrm{KK}(A, \mathbb{C}) \cong \mathrm{K}^0(A) = \text{"}\{\text{elliptic operator on } A\}\text{"},$$

- the composition (Kasparov product)

$$\otimes_{C(M)} \colon \mathrm{KK}(\mathbb{C}, C(M)) \otimes \mathrm{KK}(C(M), \mathbb{C}) \to \mathrm{KK}(\mathbb{C}, \mathbb{C}) \cong \mathbb{Z}$$

  is nothing but the index pairing $\langle [E], [D] \rangle = \mathrm{Index}(D_E)$.

A: C*-algebra.
$\mathcal{E}$: fin. gen. proj. A-module bundle over M.
Then, $[\mathcal{E}] \in \mathrm{K}_0(C(M) \otimes A)$ and the composition

$$\otimes_{C(M)} \colon \mathrm{KK}(\mathbb{C}, C(M) \otimes A) \otimes \mathrm{KK}(C(M), \mathbb{C}) \to \mathrm{KK}(\mathbb{C}, A) = \mathrm{K}_0(A)$$

is called the index pairing with coefficient in A.

The Mishchenko–Fomenko higher index

M: even dim closed (Spin^c) manifold. $\Gamma := \pi_1(M)$.
$C^*\Gamma := \overline{\mathbb{C}[\Gamma]}$: (full) group C*-algebra. Then,

$$\mathcal{E}_\Gamma^{\mathrm{MF}} := \tilde{M} \times_\Gamma C^*\Gamma$$

is a flat $C^*\Gamma$-module bundle. For a Dirac-type operator D on M,

$$\beta_M([D]) = \mathrm{Ind}_{\mathrm{MF}}\, D := [\mathcal{E}_\Gamma^{\mathrm{MF}}] \otimes_{C(M)} [D] \in \mathrm{K}_0(C^*\Gamma)$$

is called the *Mishchenko–Fomenko higher index*.

- The MF higher signature is homotopy invariant (Kaminker).
- M doesn't admit a psc metric unless the MF higher $\hat{A}$-genus vanishes.
- The MF higher index map is functorial. In particular,

$$\begin{array}{ccc} \mathrm{K}_0(M) & \xrightarrow{\beta_M} & \mathrm{K}_0(C^*\Gamma) \\ \downarrow & \nearrow_{\exists \beta} & \\ \mathrm{K}_0(B\Gamma) & & \end{array}$$

Dualizing the higher index

$$\alpha_M := [\mathcal{E}_\Gamma^{\mathrm{MF}}] \otimes_{C^*\Gamma} \cdot \; : \mathrm{KK}(C^*\Gamma, \mathbb{C}) \to \mathrm{KK}(\mathbb{C}, C(M))$$

is the adjoint of β_M, i.e. $\langle \alpha_M(x), y \rangle = \langle x, \beta_M(y) \rangle$,

$$\begin{array}{ccccc}
 & \mathrm{K}_0(C(M)) & \xleftarrow{\;\alpha_M\;} & \mathrm{K}^0(C^*\Gamma) & \ni x \\
 & \otimes & & \otimes & \\
y \ni & \mathrm{K}^0(C(M)) & \xrightarrow{\;\beta_M\;} & \mathrm{K}_0(C^*\Gamma) & \\
 & \downarrow & & \downarrow & \\
 & \mathbb{Z} & & \mathbb{Z} &
\end{array}$$

Moreover, by functoriality of α_M,

$$\begin{array}{ccc}
\mathrm{K}^0(M) & \xleftarrow{\;\alpha_M\;} & \mathrm{K}^0(C^*\Gamma) \\
\uparrow f^* & \swarrow \exists\alpha & \\
\mathrm{K}^0(B\Gamma) & &
\end{array}$$

If Γ satisfies the 'strong Novikov conjecture', $\mathrm{Im}(\alpha_M) = \mathrm{Im}(f^*)$.

K-homology of the group C*-algebra

An element of $\mathrm{K}^0(C^*\Gamma)$ is a 'difference' of two (infinite dimensional) unitary representations. i.e. $(\pi_i, \mathcal{H}_i)$ $(i = 0, 1)$ of Γ and

$$F \colon \mathcal{H}_0 \to \mathcal{H}_1 \text{ Fredholm op. s.t. } F\pi_0(g) - \pi_1(g)F \in \mathbb{K}.$$

Observation

$\alpha_M([\mathcal{H}_0, \mathcal{H}_1, F])$ *is the 'associated Fredholm bundle'*

$$"\tilde{F} \colon \tilde{M} \times_\Gamma \mathcal{H}_0 \to \tilde{M} \times_\Gamma \mathcal{H}_1"$$

This Fredholm bundle is 'flat modulo compact'.

Question

When is $\alpha_M[\mathcal{H}_0, \mathcal{H}_1, F]$ almost flat as an element of $\mathrm{K}^0(M)$?

Quasi-diagonality

Definition

A C*-algebra is *quasi-diagonal* (QD) if some (any) faithful representation π admits an increasing sequence of finite rank projections p_k such that $[p_k, \pi(a)] \to 0$ for any $a \in A$.

- All amenable group C*-algebras are QD (Tikuisis–White–Winter'16).
- A residually finite group algebra has a QD completion.

In these cases, $(\pi, \mathcal{H})$ can be approximated by infinite direct sums of finite dimensional quasi-representations.

Corollary (Dadarlat'14, K.)

If Γ *is*

- *amenable or*
- *residually finite and coarsely embeddable into Hilbert space,*

then all element in $\mathrm{K}^0(B\Gamma)$ *is almost flat.*

Relative version

Relative almost flat bundle

Let M be a compact manifold with the boundary $\partial M = N$.

Definition

An *ε-flat relative vector bundle* on (M, N) is a triple (E, E', u) where

- E and E' are hermitian vector bundles on M with the connections ∇ and ∇',
- $u \colon E|_N \to E'|_N$ is a unitary bundle isomorphism

such that

- their curvature satisfies $\|R^{\nabla}\| < \epsilon$ and $\|R^{\nabla'}\| < \epsilon$ and
- $u^* \nabla'_N u = \nabla_N$,

(where ∇_N denotes the restriction of ∇ on N.)

An element $\alpha \in \mathrm{K}^0(M, N)$ is almost flat if for any $\varepsilon > 0$ there is a ε-flat vector bundle (E, E', u) such that $\alpha = [E, E', u]$.

Chang–Weinberger–Yu relative assembly map

M: manifold with boundary $\partial M = N$.
Assume $\Lambda := \pi_1(N) \subset \pi_1(M) =: \Gamma$.

Recently, Chang–Weinberger–Yu gives a definition of

$$\begin{array}{ccccccccc} \longrightarrow & \mathrm{K}_0(N) & \longrightarrow & \mathrm{K}_0(M) & \longrightarrow & \mathrm{K}_0(M,N) & \longrightarrow \\ & \downarrow \mu_N & & \downarrow \mu_M & & \downarrow \mu_{M,N} & \\ \longrightarrow & \mathrm{K}_0(C^*\Lambda) & \longrightarrow & \mathrm{K}_0(C^*\Gamma) & \longrightarrow & \mathrm{K}_0(\text{"}\, C^*(\Gamma,\Lambda)\text{"}) & \longrightarrow \end{array}$$

For long exactness of the second row,

$$\begin{aligned} C^*(\Gamma,\Lambda) &= S\mathrm{Cone}(C^*\Lambda \to C^*\Gamma) \\ &= \left\{ f\colon [0,1]^2 \to C^*\Gamma \;\middle|\; \begin{array}{l} f(t,1) \in C^*\Lambda, \\ f(0,s) = f(1,s) = f(t,0) = 0 \end{array} \right\} \end{aligned}$$

Theorem (Deeley–Goffeng'17)

$$\mu_{M,N}([D]) = \mathrm{Index}_{\mathrm{APS}}(D,P) + \mathrm{sf}(P)$$

MF index description of the relative assembly map

Theorem (K. in preparation)

There is a 'relative Mishchenko–Fomenko bundle' $\mathcal{E}^{\mathrm{MF}}_{\Gamma,\Lambda}$*, which determines an element*

$$[\mathcal{E}^{\mathrm{MF}}_{\Gamma,\Lambda}] \in \mathrm{KK}(\mathbb{C}, C_0(M^o) \otimes C^*(\Gamma, \Lambda))$$

and the Kasparov product

$$\otimes[\mathcal{E}^{\mathrm{MF}}_{\Gamma,\Lambda}] \colon \mathrm{K}_0(M, N) \to \mathrm{K}_0(C^*(\Gamma, \Lambda))$$

coincides with the relative assembly map $\mu_{\Gamma,\Lambda}$.

By using this description of $\mu_{\Gamma,\Lambda}$, we get the dual assembly map

$$\mathrm{K}^0(C^*(\Gamma, \Lambda)) \to \mathrm{K}^0(M, N)$$

Relative quasi-representation

Theorem (K. in preparation)

An element in $\mathrm{K}_0(C^*(\Gamma, \Lambda))$ *is given by a pair* $(\pi_i, \mathcal{H}_i)$ *of unitary representations of* Γ *and a Fredholm operator* $F \colon \mathcal{H}_0 \to \mathcal{H}_1$ *s.t.*

- $F\pi_0(g) - \pi_1(g)F \in \mathbb{K}$ *for all* $g \in \Gamma$,
- $F\pi_0(g) = \pi_1(g)F$ *for all* $g \in \Lambda$.

In particular, if Γ has a QD completion, then any element of $\mathrm{K}^0(C^*(\Gamma, \Lambda))$ gives a pair of quasi-representations of Γ which are unitary equivalent after restricted to Λ.

Corollary

Let Γ *be*

- *amenable, or*
- *residually finite and coarsely embeddable into a Hilbert space,*

then all element in $\mathrm{K}^0(B\Gamma, B\Lambda)$ *is almost flat.*

Thank you!