Singularities of the $L^2$ Exponential Map on Diffeomorphism Groups

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Let $M$ be a compact, oriented Riemannian manifold of dimension $n = 2, 3$. The Cauchy problem for the Euler equations of hydrodynamics is

$$\partial_t u + \nabla_u u = -\nabla p$$
$$\text{div}(u) = 0$$
$$u(0) = u_0$$

where $u : M \times \mathbb{R} \to TM$ is the velocity field and $p : M \times \mathbb{R} \to \mathbb{R}$ is the pressure.

Remark: In Sobolev spaces, global well-posedness of (1) is known when $n = 2$ (Gunther, Lichtenstein 1920s, Wolibner, 1933).
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**Remark**

*In Sobolev spaces, global well-posedness of (1) is known when $n = 2$ (Gunther, Lichtenstein 1920s, Wolibner, 1933).*
Lagrangian description of the fluid flow on $M$:

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We denote by $\mathcal{D}_{\mu}^s(M)$ the set of all $H^s$ volume-preserving diffeomorphisms. $\mu$ denotes the Riemannian volume.
On $D^s_\mu(M)$, we have a natural Riemannian metric, the $L^2$ metric, given at the identity $e$ by

$$\langle u, v \rangle_{L^2} = \int_M \langle u(p), v(p) \rangle d\mu(p), \quad u, v \in T_eD^s_\mu(M).$$
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This metric extends smoothly by right-translations to a metric on $\mathcal{D}^s_\mu(M)$. Despite being a weak Riemannian metric, it has a Levi-Civita connection and a smooth geodesic spray (Ebin, Marsden - 1970).
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Arnold (1966) observed that a curve $\eta_t$ in $\mathcal{D}_\mu^s(M)$ is a geodesic of this metric if and only if the corresponding vector field $u(t, x)$ solves the Euler equations.
In light of Arnold’s result, $D^s_\mu(M)$ is geodesically complete when $n = 2$. 

The $L^2$ exponential map at the identity is given by 

$$\exp_e : T_{D^s_\mu(M)} \to D^s_\mu(M),$$

where $t \mapsto \eta_t$ is the unique $L^2$ geodesic with $\partial_t \eta(0, x) = u(x)$. This is a diffeomorphism near $0$.

Theorem (Ebin, Misiolek, Preston - 2006)
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**Theorem (Ebin, Misiołek, Preston - 2006)**

*The $L^2$ exponential map $\exp_e$ is a nonlinear smooth Fredholm map of index zero.*
Singularities of $\exp_e$ are called *conjugate points*. Their existence in $\mathcal{D}_\mu^s(M)$ was conjectured by Arnold and first proved by Misiołek (1993). Many other examples are known: Shnirelman 1994, when $\dim(M) \geq 3$; Misiołek 1996, when $M = T^2$ with flat metric; Ebin, Misiołek, Preston 2006, $u = \partial_\theta$ on $D_2 \times S^1$; Preston, Washabaugh 2014, axisymmetric 3D flows with swirl; Benn 2015, $\dim(M) = 2$, along isometry group.
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**Definition**

A conjugate point $u_0 \in T_e \mathcal{D}^s_\mu(M)$ is said to be regular if there exists an open set $U \subseteq T_e \mathcal{D}^s_\mu(M)$ containing $u_0$ with the following property: for any ray $\vec{r}$ intersecting $U$, the line segment $\vec{r} \cap U$ contains at most one conjugate point.
Regular Conjugate Points

$T_e D^s_\mu(M)$

$u_0$
Regular Conjugate Points

\[ T_eD_s^\mu(M) \]
Regular Conjugate Points

Let $eD^s_{\mu}(M)$ be the exponential map on the diffeomorphism group $\mathcal{D}(M)$, where $e$ is the identity element and $\mathcal{D}(M)$ is the group of diffeomorphisms of a manifold $M$. The diagram illustrates the relationship between $TeD^s_{\mu}(M)$ and the set $U$ of regular conjugate points.
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The set $C_e \subseteq T_eD^s_{\mu}(M)$ of regular conjugate points is a smooth submanifold of $T_eD^s_{\mu}(M)$ of codimension 1. Moreover, for any $u_0 \in C_e$, its tangent space satisfies

$$T_{u_0}C_e \oplus \mathbb{R}u_0 \simeq T_eD^s_{\mu}(M).$$
Regular Conjugate Points

Theorem (Smoothness of the singular set)

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$$T_{u_0}C_e \oplus \mathbb{R}u_0 \cong T_eD_s^\mu(M).$$

Main ingredients in the proof:

L2 Morse index theorem (Misiołek, Preston, 2009).

Perturbation theory of self-adjoint operators.
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Normal Forms

First, we focus on regular conjugate points of multiplicity $k \geq 2$. 

Theorem (Normal forms – First case)

Let $u_0 \in \mathbb{C} \mathbb{C}$ be a regular conjugate point of multiplicity $k \geq 2$.

Then, locally near $u_0$, $\exp$ has the form:

$$
\exp : \mathbb{R}^{k+1} \times H \to \mathbb{R}^{k+1} \times H
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where $H$ is a Hilbert space.
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\exp_e : \mathbb{R}^{k+1} \times \mathbb{H} \to \mathbb{R}^{k+1} \times \mathbb{H} \\
(t, x_1, \ldots, x_k, v) \mapsto (t, tx_1, tx_2, \ldots, tx_k, v)
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where \( \mathbb{H} \) is a Hilbert space.
Let $u_0 \in C_e$ be a regular conjugate point of multiplicity 1 such that $\ker d\exp_e(u_0) \nsubseteq T_{u_0} C_e$. Then, locally near $u_0$, $\exp_e$ has the form

$$\exp_e : \mathbb{R} \times \mathbb{H} \to \mathbb{R} \times \mathbb{H}$$

$$(t, \nu) \mapsto (t^2, \nu)$$

where $\mathbb{H}$ is a Hilbert space.
Theorem (Normal forms – Third case – cusps)

Let \( u_0 \in C_e \) be a regular conjugate point of multiplicity 1 such that \( \ker d \exp_e(u_0) \subseteq T_{u_0} C_e \). Suppose \( u_0 \) is normal to \( C_e \). Let \( \Pi \) be the \( L^2 \) shape tensor of \( C_e \subseteq T_e D^s_\mu(M) \). If

\[
\Pi(w, w) \neq -\|w\|_{L^2}^2, \quad \forall w \in \ker d \exp_e(u_0),
\]

then, locally near \( u_0 \), \( \exp_e \) has the form

\[
\exp_e : \mathbb{R}^2 \times \mathbb{H} \to \mathbb{R}^2 \times \mathbb{H}
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(t, s, v) \mapsto (t^3 - st, s, v)
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Corollary ($L^2$ Morse-Littauer)

The $L^2$ exponential map $\exp_e : T_e D^s_\mu(M) \rightarrow D^s_\mu(M)$ is not injective on any neighborhood of a conjugate point.
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Proof. First, note that all of the above local forms are not injective.
Corollary ($L^2$ Morse-Littauer)

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**Proof.** First, note that all of the above local forms are not injective. Let $u_0 \in T_e \mathcal{D}_\mu^s(M)$ be any regular conjugate point. One of the following holds:

- For all conjugate points $u$ in a neighborhood of $u_0$, we have $\ker d\exp_e(u) \subseteq T_u C_e$. 

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- There exists a sequence $\{u_n\}_{n \geq 1}$ converging to $u_0$ with $\ker d \exp_e(u_n) \nsubseteq T_{u_n} C_e$. 
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In the first case, $\exp_e$ has a normal form at $u_0$, which is not injective.
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In the first case, $\exp_e$ has a normal form at $u_0$, which is not injective.
In the second case, $\exp_e$ is a fold near each $u_n$, so it cannot be injective near $u_0$. 

The result follows from the fact that regular conjugate points are dense in the set of all conjugate points. ■
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