Singularities of the L^2 Exponential Map on Diffeomorphism Groups

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Let M be a compact, oriented Riemannian manifold of dimension $n = 2, 3$. The Cauchy problem for the Euler equations of hydrodynamics is

$$\begin{aligned} &\partial_t u + \nabla_u u = -\nabla p \\ &\mathrm{div}(u) = 0 \\ &u(0) = u_0 \end{aligned} \tag{1}$$

where $u : M \times \mathbb{R} \to TM$ is the velocity field and $p : M \times \mathbb{R} \to \mathbb{R}$ is the pressure.

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Remark

In Sobolev spaces, global well-posedness of (1) *is known when* $n = 2$ *(Gunther, Lichtenstein* 1920s*, Wolibner,* 1933*).*

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We denote by $\mathcal{D}^s_\mu(M)$ the set of all H^s volume-preserving diffeomorphisms. μ denotes the Riemannian volume.

On $\mathcal{D}^s_\mu(M)$, we have a natural Riemannian metric, the L^2 metric, given at the identity e by

$$\langle u, v\rangle_{L^2} = \int_M \langle u(p), v(p)\rangle d\mu(p), \quad u, v \in T_e\mathcal{D}^s_\mu(M).$$

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Arnold (1966) observed that a curve η_t in $\mathcal{D}^s_\mu(M)$ is a geodesic of this metric if and only if the corresponding vector field $u(t,x)$ solves the Euler equations.

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The L^2 exponential map at the identity is given by

$$\begin{aligned} \exp_e : T_e\mathcal{D}^s_\mu(M) &\to \mathcal{D}^s_\mu(M) \\ u &\mapsto \eta_1, \end{aligned}$$

where $t \mapsto \eta_t$ is the unique L^2 geodesic with $\partial_t \eta(0, x) = u(x)$. This is a diffeomorphism near 0.

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Theorem (Ebin, Misiołek, Preston - 2006)

The L^2 exponential map $\exp_e$ is a nonlinear smooth Fredholm map of index zero.

Singularities of $\exp_e$ are called *conjugate points*. Their existence in $\mathcal{D}^s_\mu(M)$ was conjectured by Arnold and first proved by Misiołek (1993).

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- Benn 2015, $\dim(M) = 2$, along isometry group.

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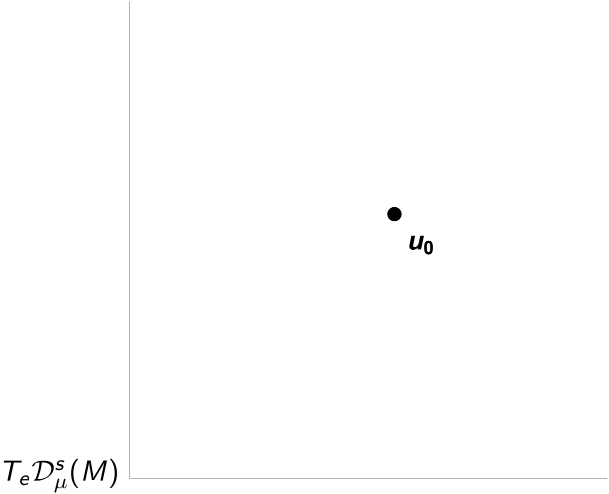
Definition

A conjugate point $u_0 \in T_e\mathcal{D}^s_\mu(M)$ is said to be **regular** *if there exists an open set $U \subseteq T_e\mathcal{D}^s_\mu(M)$ containing u_0 with the following property: for any ray $\vec{r}$ intersecting U, the line segment $\vec{r} \cap U$ contains at most one conjugate point.*

Regular Conjugate Points



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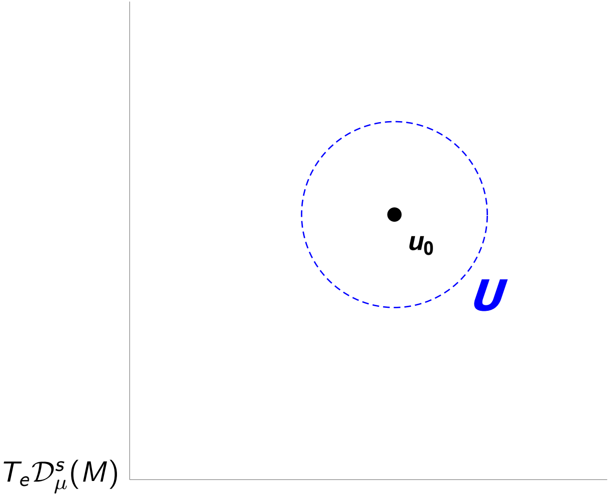

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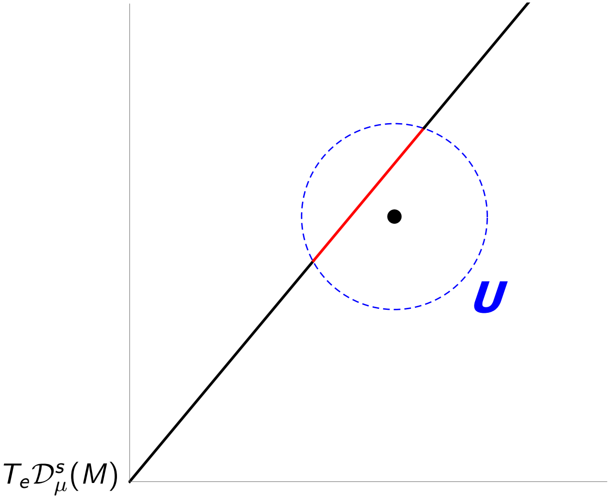

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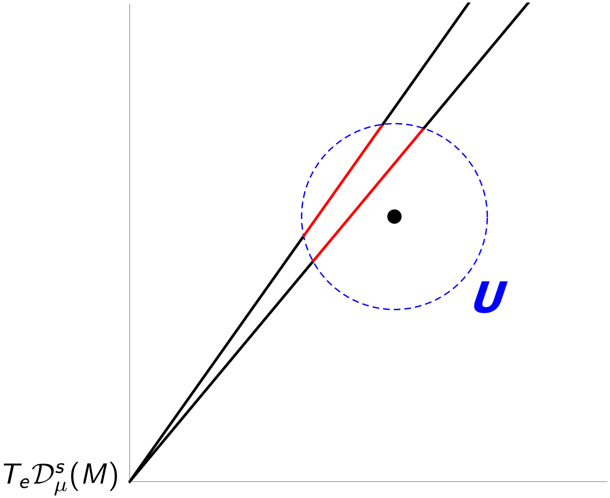

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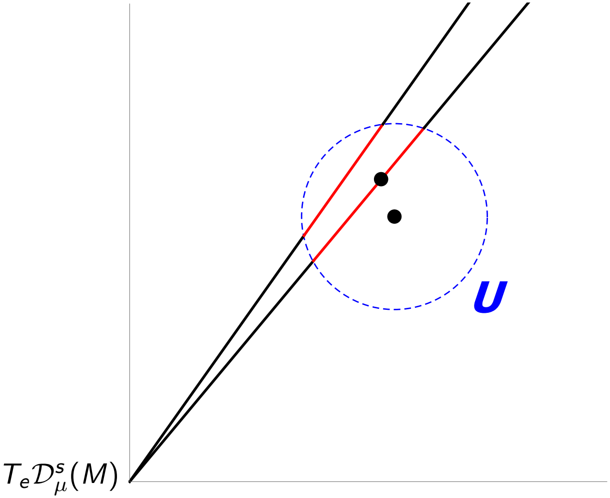

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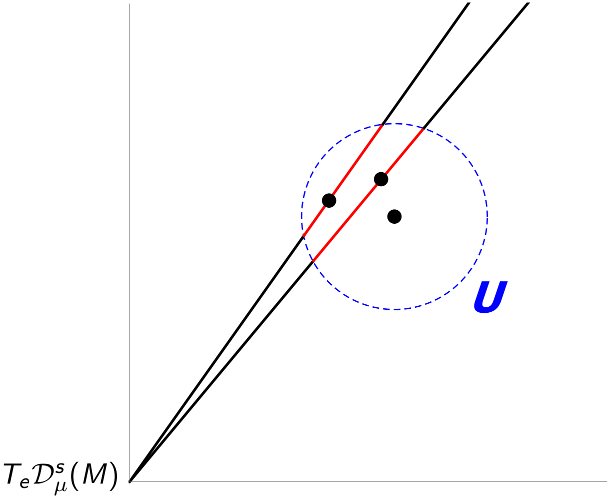

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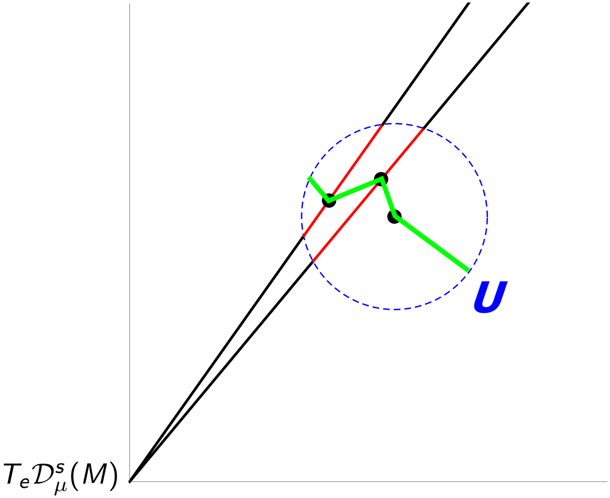

Regular Conjugate Points

Theorem (Smoothness of the singular set)

The set $C_e \subseteq T_e\mathcal{D}^s_\mu(M)$ of regular conjugate points is a smooth submanifold of $T_e\mathcal{D}^s_\mu(M)$ of codimension 1. Moreover, for any $u_0 \in C_e$, its tangent space satisfies

$$T_{u_0}C_e \oplus \mathbb{R}u_0 \simeq T_e\mathcal{D}^s_\mu(M).$$

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Main ingredients in the proof:

- L^2 Morse index theorem (Misiołek, Preston, 2009).
- Perturbation theory of self-adjoint operators.

Normal Forms

First, we focus on regular conjugate points of multiplicity $k \geq 2$.

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Theorem (Normal forms – First case)

Let $u_0 \in C_e$ be a regular conjugate point of multiplicity $k \geq 2$. Then, locally near u_0, $\exp_e$ has the form

$$\begin{aligned} \exp_e : \mathbb{R}^{k+1} \times \mathbb{H} &\to \mathbb{R}^{k+1} \times \mathbb{H} \\ (t, x_1, \ldots, x_k, v) &\mapsto (t, tx_1, tx_2, \ldots, tx_k, v) \end{aligned}$$

where $\mathbb{H}$ is a Hilbert space.

Normal Forms

Theorem (Normal forms – Second case – folds)

Let $u_0 \in C_e$ *be a regular conjugate point of multiplicity* 1 *such that* $\ker d\exp_e(u_0) \not\subseteq T_{u_0}C_e$*. Then, locally near* u_0, $\exp_e$ *has the form*

$$\begin{aligned} \exp_e : \mathbb{R} \times \mathbb{H} &\to \mathbb{R} \times \mathbb{H} \\ (t, v) &\mapsto (t^2, v) \end{aligned}$$

where $\mathbb{H}$ *is a Hilbert space.*

Normal Forms

Theorem (Normal forms – Third case – cusps)

Let $u_0 \in C_e$ be a regular conjugate point of multiplicity 1 such that $\ker d\exp_e(u_0) \subseteq T_{u_0}C_e$. Suppose u_0 is normal to C_e. Let Π be the L^2 shape tensor of $C_e \subseteq T_e\mathcal{D}^s_\mu(M)$. If

$$\Pi(w,w) \neq -\|w\|^2_{L^2}, \ \forall w \in \ker d\exp_e(u_0),$$

then, locally near u_0, $\exp_e$ has the form

$$\begin{aligned} \exp_e : \mathbb{R}^2 \times \mathbb{H} &\to \mathbb{R}^2 \times \mathbb{H} \\ (t,s,v) &\mapsto (t^3 - st, s, v) \end{aligned}$$

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Normal Forms

Corollary (L^2 Morse-Littauer)

The L^2 exponential map $\exp_e : T_e\mathcal{D}^s_\mu(M) \to \mathcal{D}^s_\mu(M)$ is not injective on any neighborhood of a conjugate point.

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Proof. First, note that all of the above local forms are not injective.

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Proof. First, note that all of the above local forms are not injective. Let $u_0 \in T_e\mathcal{D}^s_\mu(M)$ be any regular conjugate point. One of the following holds:

- For all conjugate points u in a neighborhood of u_0, we have $\ker d\exp_e(u) \subseteq T_u C_e$.

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In the second case, $\exp_e$ is a fold near each u_n, so it cannot be injective near u_0.

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The result follows from the fact that regular conjugate points are dense in the set of all conjugate points. ■