"Twisted Alexander polynomials of hyperbolic links and a conjecture of Dunfield, Friedl and Jackson" (with A. Tran)

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§ Introduction

$K \subset S^3$ a knot, $G(K) := \pi_1 E(K)$ the knot group, $E(K) := S^3 \setminus \mathring{N}(K)$ the exterior of K

$\rho : G(K) \to GL(m, \mathbb{F})$ a representation, $\mathbb{F}$: a field

$\Rightarrow$ the twisted Alexander polynomial (TAP) $\Delta_{K,\rho}(t)$ is defined (Lin, Wada)

* If $\rho = 1$ the trivial 1-dim rep. $\Rightarrow \Delta_{K,1}(t) = \dfrac{\Delta_K(t)}{t-1}$

Suppose K is hyperbolic (i.e. $S^3 \setminus K$ has the complete hyp. metric of finite volume)

$\rho_0 : G(K) \to SL(2, \mathbb{C})$ a lift of the holonomy rep. $\bar{\rho}_0 : G(K) \to PSL(2, \mathbb{C})$

$\mathcal{T}_K(t) \overset{def}{:=} \Delta_{K,\rho_0}(t) \in \mathbb{C}[t^{\pm 1}]$ the hyp. torsion poly. $(\mathcal{T}_K(t^{-1}) = \mathcal{T}_K(t))$

* $g(K)$:= the min. genus of Seifert surfaces of K.
* K : fibered $\overset{def}{\Longleftrightarrow}$ $E(K)$ fibers over the circle.

Conj 1 (DFJ) For a hyp. knot K in S^3

(A) $\mathcal{T}_K(t)$ determines $g(K)$ i.e. $\deg \mathcal{T}_K(t) = 4g(K) - 2$ holds

(B) K is fibered iff $\mathcal{T}_K(t)$ is monic (i.e. the coeff. of the highest degree term is one)

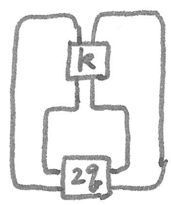
Rmk ① For a 2-dim rep, $\deg \Delta_{K,\rho}(t) \leq 4g(K) - 2$ (Friedl-Kim)

② $\{\Delta_{K,\rho}(t) \mid \rho : G(K) \to GL(m, \mathbb{C})\}$ determines $g(K)$ & fiberedness of K (Friedl-Vidussi)

③ Conj 1 is true for all hyp. knots in S^3 with 15 or fewer crossings (313,209 knots) [DFJ]

Thm 1 (M-Tran)

Let $K = J(k, 2q)$ the double twist knot.
Then Conj 1 is true.

$J(k, 2q) =$ 

$k, 2q$ = # of half twists ($k > 0$ right-handed, $k < 0$ left-handed)

Rmk Recently Agol-Dunfield showed Conj 1(A) for "libroid" hyp knots in S^3.

* We generalize Conj 1 to an oriented hyp link in S^3 and verify it for an infinite family of hyp 2-component links.

§ Review of TAP

$G(K) = \langle x_1, \cdots, x_n \mid r_1, \cdots, r_{n-1} \rangle$ Fix a deficiency one pres. (e.g. Wirtinger pres.)

$\left.\begin{array}{l} \alpha_K : G(K) \to H_1(E(K); \mathbb{Z}) \cong \langle t \rangle \text{ abelianization} \\ \rho : G(K) \to SL(2, \mathbb{C}) \text{ a rep.} \end{array}\right\} \Rightarrow (\alpha_K \otimes \rho)(x) := \alpha_K(x)\rho(x), \ x \in G(K)$

$\Phi : \mathbb{Z}[F_n] \xrightarrow{\text{pres.}} \mathbb{Z}[G(K)] \xrightarrow{\tilde{\alpha}_K \otimes \tilde{\rho}} M(2, \mathbb{C}[t^{\pm 1}])$

$A = (a_{ij})$: $2(n-1) \times 2n$ matrix, $a_{ij} = \Phi\left(\frac{\partial r_i}{\partial x_j}\right)$, $\frac{\partial}{\partial x_j} : \mathbb{Z}[F_n] \to \mathbb{Z}[F_n]$ Fox derivative

A_j : $2(n-1) \times 2(n-1)$ matrix (remove the jth column) $\in M(n-1, M(2, \mathbb{C}[t^{\pm 1}])) = M(2(n-1), \mathbb{C}[t^{\pm 1}])$

Def (TAP) "Wada inv"

$$\Delta_{K,\rho}(t) := \frac{\det A_j}{\det \Phi(x_j - 1)} \quad \text{rational function}$$

well-def up to multiplication by t^{2i} ($i \in \mathbb{Z}$)

Rmk ① If $\rho \sim \rho'$ conjugate $\Rightarrow \Delta_{K,\rho}(t) = \Delta_{K,\rho'}(t)$

② If $\rho : G(K) \to SL(2, \mathbb{C})$ nonabelian $\Rightarrow \Delta_{K,\rho}(t) \in \mathbb{C}[t^{\pm 1}]$ (Kitano-M)

③ $\Delta_{K,\rho}(t^{-1}) = t^i \Delta_{K,\rho}(t)$ for some $i \in \mathbb{Z}$ "reciprocal" (Hillman-Silver-Williams)

Ex. $K = 4_1$ (the figure eight knot) genus one & fibered

$G(K) = \langle x_1, x_2 \mid r = w x_1 w^{-1} x_2^{-1} \rangle$, $w = [x_2, x_1^{-1}]$

$\rho : G(K) \to SL(2, \mathbb{C})$ a rep. defined by

$\rho(x_1) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $\rho(x_2) = \begin{pmatrix} 1 & 0 \\ -\omega & 1 \end{pmatrix}$, $\omega \in \mathbb{C}$ s.t. $1 + \omega + \omega^2 = 0$

$$\Rightarrow \Delta_{K,\rho}(t) = \frac{\det \Phi\left(\frac{\partial r}{\partial x_1}\right)}{\det \Phi(x_2 - 1)} = \frac{t^{-2}(t-1)^2(t^2 - 4t + 1)}{(t-1)^2} \doteq t^2 - 4t + 1$$

Rmk. In general, for a fibered knot K, $\Delta_{K,\rho}(t)$ is monic (Cha, Goda-Kitano-M) and of degree $4g(K) - 2$ (Kitano-M).

§ DFJ conjecture

N: a complete hyp 3-mfd of finite volume $\Rightarrow$ $\exists$ discrete faithful rep. $\bar{\rho}_0 : \pi_1 N \to PSL(2,\mathbb{C}) \cong \mathrm{Isom}^+(\mathbb{H}^3)$ s.t. $\mathbb{H}^3/\mathrm{Im}\bar{\rho}_0 \cong N$, called the holonomy rep, unique up to conj.

DFJ studied for $N = E(K)$, $K \subset S^3$ a hyp knot and $\rho_0 : G(K) \to SL(2,\mathbb{C})$ a lift of $\bar{\rho}_0$ s.t. $\mathrm{tr}\,\rho_0(\mu_K) = 2$ (meridian of K)

* A generalization

$L = L_1 \cup \cdots \cup L_\ell \subset S^3$ an "oriented" hyp. link

$\bar{\rho}_0 : G(L) \to PSL(2,\mathbb{C})$ the holonomy rep, $\exists\, 2^\ell$ lifts to $SL(2,\mathbb{C})$

We focus on the lift $\rho_0 : G(L) \to SL(2,\mathbb{C})$ s.t. $\mathrm{tr}\,\rho_0(\mu_i) = 2$ $(i = 1, \cdots, \ell)$, μ_i : a meridian of L_i

Conj 2 For an oriented hyp "alternating" link $L = L_1 \cup \cdots \cup L_\ell \subset S^3$,

(A) TAP determines $g(L)$ i.e. $\underbrace{\deg \Delta^{\alpha}_{L,\rho_0}(t) = 4g(L) + 2(\ell - 2)}_{(*)}$, $\alpha : G(L) \longrightarrow \mathbb{Z} = \langle t \rangle$, $\mu_i \longmapsto t$

(B) L is fibered iff $\Delta^{\alpha}_{L,\rho_0}(t)$ is monic

Rmk For a general hyp link L, $(*)$ will be $\deg \Delta^{\alpha}_{L,\rho_0}(t) = 2\|\alpha\|_T$ the "Thurston norm" of $\alpha \in H^1(E(L); \mathbb{Z})$ as above.

Thm 2 (M-Tran)

For the double twist link $L = J(2m+1, 2n+1)$ with $m, n \notin \{-1, 0\}$ & any orientation, Conj 2 is true.

Rmk Conj 1 and 2 are still widely open.