Relations between abstract patterns and the corresponding dynamical systems

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Characters of the talk

Delone sets ← first part of the talk

Delone multi sets

tilings

functions

certain measures (such as $\sum g(x)\delta_x$)

second part of the talk

these are not necessarily periodic, but "close to" periodic

(several interpretations for "close to")

Outline of the talk

first half: On Delone sets and the corresponding dynamical systems

last half: On a general treatment of characters in the previous slide

What are Delone sets?

X: a metric space

$D \subset X$ is called a *Delone set* if it is

relatively dense in X

and

uniformly discrete

What are Delone sets?

X: a metric space

$D \subset X$ is *relatively dense* in X if

$$\exists R > 0 \text{ such that } D \cap B(x, R) \neq \emptyset \text{ for any } x \in X.$$

Example

$$\text{lattices in } X = \mathbb{R}^d \text{ are relatively dense in } X.$$

Non-example

$$\{\pm 2^n \mid n = 1, 2, 3, \ldots\} \text{ is not.}$$

What are Delone sets?

X: a metric space

$D \subset X$ is *uniformly discrete* if

$$\inf_{x,y \in D, x \neq y} \rho(x,y) > 0.$$

Example

lattices in $X = \mathbb{R}^d$ are uniformly discrete.

Non-example

$\{n + 1/n \mid n \in \mathbb{Z} \setminus \{0\}\} \cup \mathbb{Z}$ is not.

What are Delone sets?

X: a metric space

$D \subset X$ is called a *Delone set* if it is

relatively dense in X

$(\exists R > 0$ such that $D \cap B(x, R) \neq \emptyset$ for any $x \in X)$

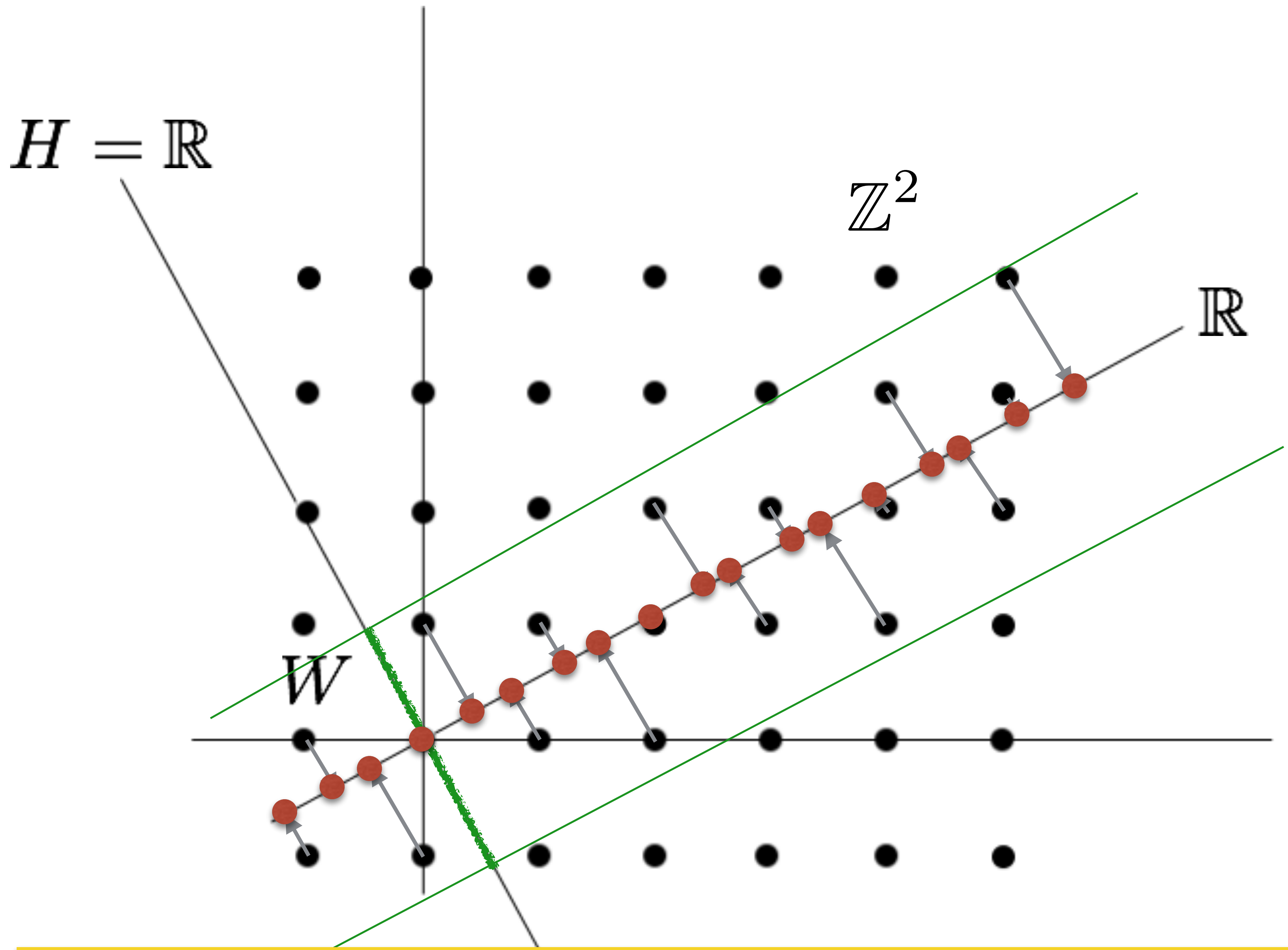
and

uniformly discrete $(\inf_{x,y \in D, x \neq y} \rho(x, y) > 0)$

Example

lattices in $X = \mathbb{R}^d$ are Delone sets.

An example of non-periodic Delone set



non-periodic, but “close to” periodic

Dynamical systems for Delone sets

D: a Delone set in $\mathbb{R}^d$

$\Omega_D = \overline{\{D + x \mid x \in \mathbb{R}^d\}}$

:closure with respect to a "local" topology

Remark

- Often Ω_D compact (henceforth assume this)
- $\forall D' \in \Omega_D$ is a Delone set

Dynamical systems for Delone sets

D: a Delone set in $\mathbb{R}^d$

$$\Omega_D = \overline{\{D + x \mid x \in \mathbb{R}^d\}}$$

:closure with respect to a "local" topology

$$\Omega_D \times \mathbb{R}^d \ni (D', x) \mapsto D' + x \in \Omega_D$$

:continuous action $\mathbb{R}^d \curvearrowright \Omega_D$

Dynamical systems for Delone sets

$\Omega_D \times \mathbb{R}^d \ni (D', x) \mapsto D' + x \in \Omega_D$

:continuous action $\mathbb{R}^d \curvearrowright \Omega_D$

$\rightsquigarrow$ collection of operators $\alpha_x \colon C(\Omega_D) \to C(\Omega_D)$

$(\alpha_x(f)(D') = f(D' - x))$

$f \in C(\Omega_D) \setminus \{0\}$:simultaneously eigenfunction for all α_x

$\rightsquigarrow$ f is called an eigenfunction

for the dynamical system $(\Omega_D, \mathbb{R}^d)$

Dynamical systems for Delone sets

$f \in C(\Omega_D) \setminus \{0\}$:simultaneously eigenfunction for all α_x

$\rightsquigarrow f$ is called an eigenfunction

for the dynamical system $(\Omega_D, \mathbb{R}^d)$

$\mathbb{R}^d \ni x \mapsto$ the eigenvalue for α_x

is a continuous character for $\mathbb{R}^d$ called eigencharacter

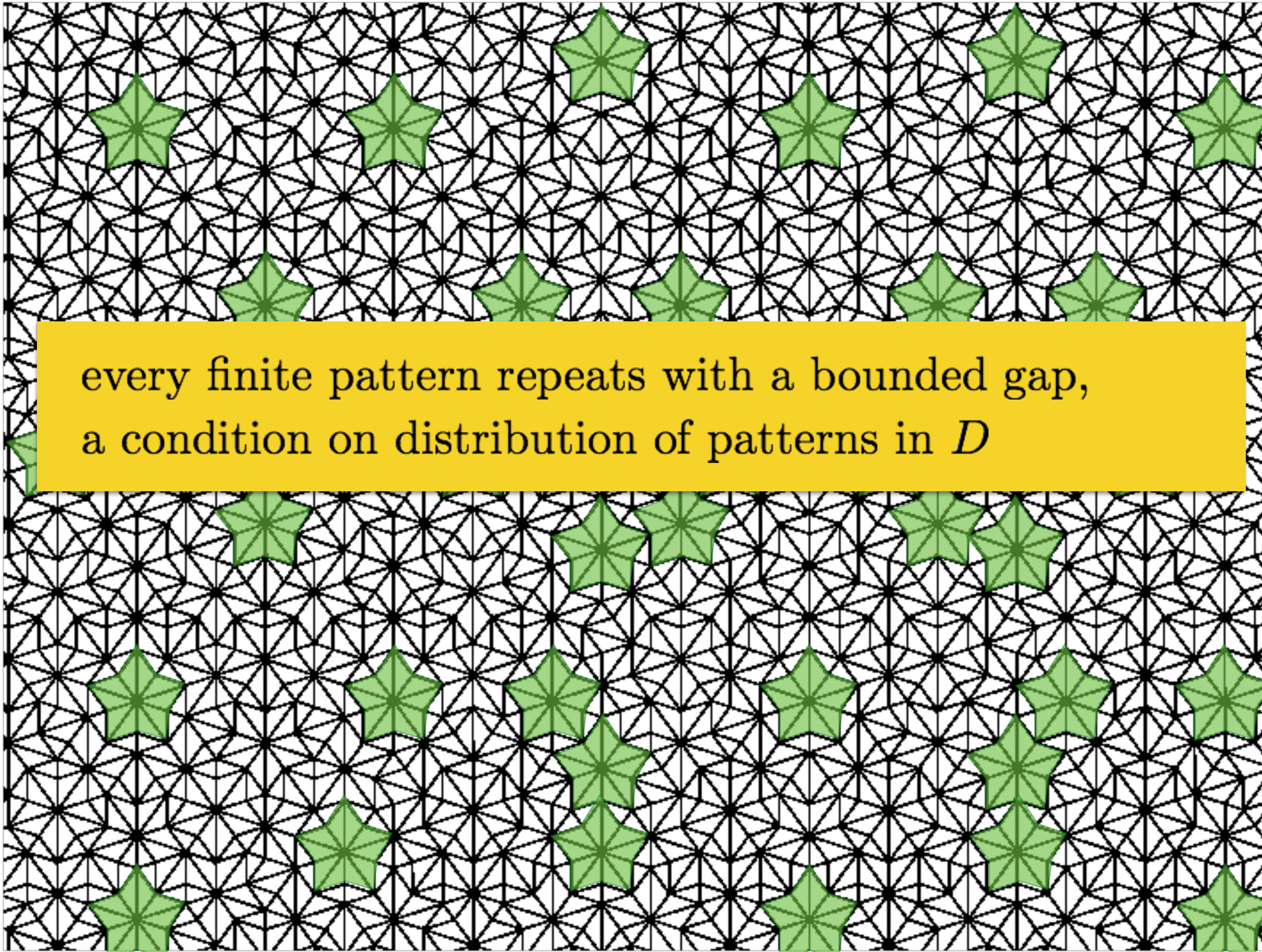
(an element of $(\mathbb{R}^d)\hat{} = \mathbb{R}^d$)

A known result

D: weakly repetitive $\iff$ $(\Omega_D, \mathbb{R}^d)$ minimal

every orbit is dense,
a condition on the dynamical system

every finite pattern repeats with a bounded gap,
a condition on distribution of patterns in D



every finite pattern repeats with a bounded gap,
a condition on distribution of patterns in D

A new result(N)

D: weakly repetitive

Then the following two conditions are equivalent:

(1) $0 \in \mathbb{R}^d$ is a limit point of the set of all eigencharacters.

(a condition on the dynamical system)

(2) for any $R_1, R_2 > 0$ and $\varepsilon > 0$, there are $L_1, L_2 > 0$

such that

(i) $|R_i - L_i| < \varepsilon$ for each $i = 1, 2$, and

(ii) D admits (L_1, L_2)-stripe structure.

(a condition on the distribution of patterns)

A new result(N)

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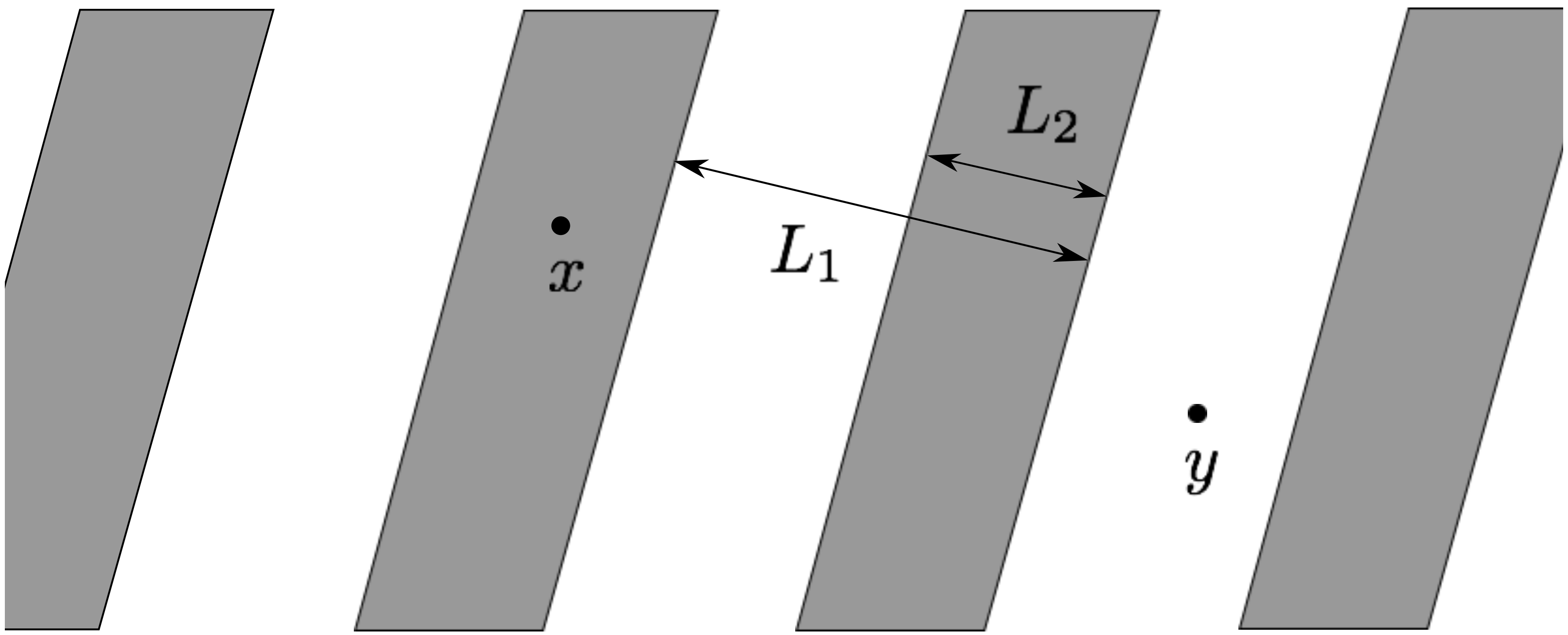
i.e. there exist $a \in \mathbb{R}^d$ with $\|a\| = 1$ and $R > 0$ such that,

$$\langle x - y, a \rangle \notin L_1\mathbb{Z} + [-\frac{1}{2L_2}, \frac{1}{2L_2}]$$

$$\Rightarrow (D - x) \cap B(0, R) \neq (D - y) \cap B(0, R)$$

A new result(N)



Outline of the talk

first half: On Delone sets and the corresponding dynamical systems

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The operation of cutting off

X: a metric space

$\mathcal{C}(X)$: the set of all closed subsets of X

$\mathrm{UD}(X)$: the set of all uniformly discrete subsets of X

the operation of “cutting off”:

$$\mathrm{UD}(X) \times \mathcal{C}(X) \ni (D, C) \mapsto D \cap C \in \mathrm{UD}(X)$$

The operation of cutting off

the operation of “cutting off”:

$$\mathrm{UD}(X) \times \mathcal{C}(X) \ni (D, C) \mapsto D \cap C \in \mathrm{UD}(X)$$

(1) for any $D \in \mathrm{UD}(X)$ and $C_1, C_2 \in \mathcal{C}(X)$, we have

$$D \cap (C_1 \cap C_2) = (D \cap C_1) \cap C_2$$

(2) for any D there is $C_D \in \mathcal{C}(X)$ such that,

$$D \cap C = D \iff C \supset C_D$$

for any $C \in \mathcal{C}(X)$.

The operation of cutting off

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(2) for any D there is $C_D \in \mathcal{C}(X)$ such that,

$$D \cap C = D \iff C \supset C_D$$

for any $C \in \mathcal{C}(X)$.

(3) if there is a continuous group action $\Gamma \curvearrowright X$,

$$\gamma(D \cap C) = (\gamma D) \cap (\gamma C)$$

for any $D \in \mathrm{UD}(X), C \in \mathcal{C}(X), \gamma \in \Gamma$.

Recall: Characters of the talk

Delone sets

Delone multi sets

tilings

functions

certain measures (such as $\sum g(x)\delta_x$)

Characters of the talk

the set of...

Delone sets $\subset$ the set of all uniformly discrete sets

Delone multi sets

$\subset$ the set of all tuples of uniformly discrete sets

tilings $\subset$ the set of all patches

functions

certain measures (such as $\sum g(x)\delta_x$)

Characters of the talk

the set of all uniformly discrete sets

the set of all tuples of uniformly discrete sets

the set of all patches

the set of functions

the set of certain measures (such as $\sum g(x)\delta_x$)

These admits an operation of “cutting off”

with the three conditions satisfied

$\rightarrow$the three conditions become the axiom

The definition of pattern space

X: a metric space

$\mathcal{C}(X)$: the set of all closed subsets of X

continuous group action $\Gamma \curvearrowright X$

Definition

A non-empty set Π is called a pattern space over (X, Γ)

if there is an operation

$$\Pi \times \mathcal{C}(X) \ni (\mathcal{P}, C) \mapsto \mathcal{P} \cap C \in \Pi$$

and an action $\Gamma \curvearrowright \Pi$

with the three conditions.

The definition of pattern space

Definition

A non-empty set Π is called a pattern space over (X,Γ) if there is an operation

$$\Pi \times \mathcal{C}(X) \ni (\mathcal{P}, C) \mapsto \mathcal{P} \cap C \in \Pi$$

and an action $\Gamma \curvearrowright \Pi$ such that

(1) $(\mathcal{P} \cap C_1) \cap C_2 = \mathcal{P} \cap (C_1 \cap C_2)$
for any $\mathcal{P} \in \Pi$ and $C_1, C_2 \in \mathcal{C}(X)$

(2) for any $\mathcal{P} \in \Pi$ there is $C_{\mathcal{P}} \in \mathcal{C}(X)$ such that

$$\mathcal{P} \cap C = \mathcal{P} \iff C \supset C_{\mathcal{P}}$$

(3) $\gamma(\mathcal{P} \cap C) = (\gamma\mathcal{P}) \cap (\gamma C)$
for any $\mathcal{P} \in \Pi$, $C \in \mathcal{C}(X)$ and $\gamma \in \Gamma$

Examples of pattern space

the set of all uniformly discrete sets

$\supset$ the subshift of all Delone sets

the set of all tuples of uniformly discrete sets

$\supset$ the subshift of all Delone multi sets

the set of all patches

$\supset$ the subshift of all tilings

the set of functions

the set of certain measures (such as $\sum g(x)\delta_x$)

Recall: a known result

D: weakly repetitive $\iff$ $(\Omega_D, \mathbb{R}^d)$ minimal

every orbit is dense,
a condition on the dynamical system

every finite pattern repeats with a bounded gap,
a condition on distribution of patterns in D

A new result(N)

$\mathbb{R}^d < \Gamma < \mathbb{R}^d \rtimes \mathrm{O}(d)$: a closed subgroup

$\mathcal{P}$: an element of a pattern space over $(\mathbb{R}^d, \Gamma)$

(with a mild assumption)

$\mathcal{P}$: weakly repetitive $\iff$ $(\Omega_{\mathcal{P}}, \Gamma)$ minimal

every orbit is dense,
a condition on the dynamical system

every finite pattern repeats with a bounded gap,
a condition on distribution of patterns in D

Recall: a result by the author

D: weakly repetitive Delone set in $\mathbb{R}^d$

Then the following two conditions are equivalent:

(1) $0 \in \mathbb{R}^d$ is a limit point of the set of all eigencharacters.

(a condition on the dynamical system)

(2) for any $R_1, R_2 > 0$ and $\varepsilon > 0$, there are $L_1, L_2 > 0$

such that

(i) $|R_i - L_i| < \varepsilon$ for each $i = 1, 2$, and

(ii) D admits (L_1, L_2)-stripe structure.

(a condition on the distribution of patterns)

Main result

$\mathcal{P}$: an element of a pattern space over $(\mathbb{R}^d, \mathbb{R}^d)$

(with a mild assumption)

Then the following two conditions are equivalent:

(1) $0 \in \mathbb{R}^d$ is a limit point of the set of all eigencharacters.

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(i) $|R_i - L_i| < \varepsilon$ for each $i = 1, 2$, and

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(a condition on the distribution of patterns)

Thank you for your attention.