K-amenability and the Baum-Connes Conjecture for a-T-menable groups

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I. Objective

This poster aims to describe the Higson-Kasparov Theorem which proves K-amenability and the Baum-Connes Conjecture for so called a-T-menable groups: groups which act properly and isometrically on Hilbert space.

II. Notation and Terminology

- In this poster, G always denotes a countable discrete group.
- By a representation of a group G, we mean a complex (separable) Hilbert space $\mathcal{H}$ equipped with a group homomorphism from G to the group $U(\mathcal{H})$ of unitary operators on $\mathcal{H}$.
- A left regular representation λ_G of G is a Hilbert space $l^2(G)$ of square summable functions on G equipped with a left translation action of G.
- Given representations of G on Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, $\mathcal{H}_1$ is said to be weakly contained in $\mathcal{H}_2$, if for each $\xi \in \mathcal{H}_1$, for any finite subset F of G and for any $\epsilon > 0$, there are vectors ξ_j $(j = 1, \cdots, m)$ in $\mathcal{H}_2$ such that $|\sum_{j=1}^{m} \langle \xi_j, g(\xi_j) \rangle_{\mathcal{H}_2} - \langle \xi, g(\xi) \rangle_{\mathcal{H}_1}| < \epsilon$ for all $g \in F$.

III. Amenability of groups

The notion of amenability of groups offers us a rich field of study where Geometry, Analysis and Representation Theory of Groups intersect each other.

Definition. A group G is **amenable** if a trivial representation of G on a one-dimensional Hilbert space $\mathbb{C}$ is weakly contained in the left regular representation λ_G of G on $l^2(G)$.

Example.
- Any finite group is amenable;
- Any abelian group is amenable, e.g. $G = \mathbb{Z}$;
- Any non-abelian free group F_n is not amenable.

Finite groups are the only trivial examples where the trivial representation of G on $\mathbb{C}$ is actually contained in the left regular representation λ_G of G on $l^2(G)$.

IV. Kasparov's group ring $R(G) = KK^G(\mathbb{C}, \mathbb{C})$

Kasparov's equivariant KK-theory allows us to study the representations of a group G in the framework of Index theory.

Definition. Kasparov's group ring $R(G)$ of a group G is a ring consisting of equivalence classes of cycles of the form $[\mathcal{H}_0 \xrightarrow{U} \mathcal{H}_1]$ where $\mathcal{H}_0, \mathcal{H}_1$ are (complex separable) Hilbert spaces equipped with representations of G and U is a bounded linear operator where $\mathrm{Id}_{\mathcal{H}_0} - U^*U$, $\mathrm{Id}_{\mathcal{H}_1} - UU^*$ and $g(U) - U$ are all compact operators.

- Equivalence relations are defined by introducing the notion of homotopy of such cycles;
- Cycles of the form $[\mathcal{H} \xrightarrow{\mathrm{Id}_{\mathcal{H}}} \mathcal{H}]$ defines zero element in the ring;
- Addition is defined by means of direct sums of cycles;
- Multiplication is defined by means of Kasparov product;
- The cycle $[\mathbb{C} \xrightarrow{0} 0]$ where a representation of G on $\mathbb{C}$ is trivial defines the identity element of the ring and denoted as 1_G.

V. K-amenability of groups

The notion of K-amenability was first introduce by J. Cuntz in 1983 to study the K-theoretical amenability of groups.

Definition. A group G is K**-amenable** if the identity 1_G in $R(G)$ can be expressed as a cycle of the form $[\mathcal{H}_0 \xrightarrow{U} \mathcal{H}_1]$ where $\mathcal{H}_0, \mathcal{H}_1$ are weakly contained in the left regular representation of G.

Example.
- Any amenable group G is K-amenable;
- Any group which acts properly on a tree is K-amenable;
- Any non-compact group having Kazhdan's property-(T) is not K-amenable.

VI. a-T-menable groups

Definition. A group G is **a-T-menable** if it acts properly and affine isometrically on a (real) Hilbert space.

Example.
- Any amenable group G is a-T-menable;
- Any group which acts properly on a tree is a-T-menable;
- Any non-compact group having Kazhdan's property-(T) is not a-T-menable.

Theorem. (Higson Kasparov, 2001) All a-T-menable groups are K-amenable.

VII. The Baum-Connes Conjecture

Interestingly, the K-amenability of a-T-menable groups was proved as a byproduct of proving the Baum-Connes Conjecture for such groups.

The Baum-Connes Conjecture (Baum Connes, 1982)

- (Conjecture): The assembly map $\mu_G : K_*^G(EG) \to K_*(C_r^*(G))$ is an isomorphism of abelian groups for any group G.
- LH: Equivariant K-homology group of classifying space EG.
- RH: K-theory group of the reduced group C^*-algebra of G.
- The assembly map μ_G is defined by taking "G-Index" of (abstract) G-equivariant Dirac operators.
- It has been verified for example, for any a-T-menable groups, for any hyperbolic groups and for any Lie groups.
- The general case (e.g. $G = SL(n, \mathbb{Z})$ $(n \geq 3)$) is still open.

VIII. The Higson-Kasparov Theorem

Theorem. (Higson Kasparov 2001) All a-T-menable groups are K-amenable and satisfy the Baum-Connes Conjecture.

- The proof of this theorem follows the standard method to prove the Baum-Connes Conjecture, so-called Dual-Dirac method.
- The method applied to an a-T-menable group G shows the identity 1_G in $R(G)$ can be expressed as geometrically nice as possible form which implies the Baum-Connes Conjecture.
- For example, it follows 1_G can be expressed as a cycle of the form $[\mathcal{H}_0 \xrightarrow{U} \mathcal{H}_1]$ where $\mathcal{H}_0, \mathcal{H}_1$ is **contained** in the (amplified) left regular representation of G.
- This is a-priori stronger than saying the K-amenability of G and highly-nontrivial even for amenable groups.

Summary

(i). The K-amenability of groups was introduced to study K-theoritic amenability of groups.
(ii). The K-amenability is somehow tightly related to the Baum-Connes Conjecture.
(iii). The Higson-Kasparov Theorem shows the K-amenability and the Baum-Connes Conjecture for all a-T-menable groups.