K-amenability and the Baum-Connes Conjecture for groups acting on trees

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Introduction

Suppose we have a group G which acts properly on a tree, e.g. Integer $\mathbb{Z}$, free groups F_n, $SL(2, \mathbb{Z})$, $SL(2, \mathbb{Q}_p)$...

What can we say about the group G?

- K-amenability (Julg and Valette, 1984);
- Baum-Connes Conjecture (Kasparov and Skandalis, 1991).

Plan of the talk

Main Topic: K-amenability for groups acting on trees

- What does it mean for groups to be K-amenable?
- (Briefly) What did Julg and Valette do?

Amenability of groups

We are interested in unitary representations of groups.

- G : a countable discrete group;

Definition

A group G is **amenable** if the trivial representation of G on $\mathbb{C}$ is **weakly contained** in the left regular representation of G on $l^2(G)$.

- Any finite group is amenable;
- Any abelian group is amenable, e.g. $G = \mathbb{Z}$;
- Any non-abelian free group F_n is not amenable.

Finite groups are the only trivial examples where
the trivial representation of G on $\mathbb{C}$ is actually
contained in the left regular representation of G on $l^2(G)$.

Kasparov's group ring $R(G) = KK^G(\mathbb{C}, \mathbb{C})$

- Kasparov's ring $R(G) := \{[\mathcal{H}_0 \xrightarrow{U} \mathcal{H}_1]\}$ / homotopy;
- $\mathcal{H}_0, \mathcal{H}_1$ are Hilbert spaces with representations of G;
- $\mathrm{Id}_{\mathcal{H}_0} - U^*U$, $\mathrm{Id}_{\mathcal{H}_1} - UU^*$ are compact (Fredholm propetrty);
- $g(U) - U$ is compact ("essentially" equivariant);
- Cycles $[\mathcal{H} \xrightarrow{\mathrm{Id}_{\mathcal{H}}} \mathcal{H}]$ are 0;
- Addition is direct sum;
- Multiplication is Kasparov product;
- The cycle $[\mathbb{C} \xrightarrow{0} 0]$ with the trivial representation of G on $\mathbb{C}$ defines the identity element 1_G of the ring.

K-amenability

The K-amenability of groups was first introduced by J. Cuntz in 1983 to study the K-theoretical amenability of groups.

Definition

A group G is ***K*-amenable** if the identity 1_G in $R(G)$ can be expressed as a cycle of the form $[\mathcal{H}_0 \xrightarrow{U} \mathcal{H}_1]$ where $\mathcal{H}_0, \mathcal{H}_1$ are **weakly contained** in the left regular representation of G.

- Any amenable group G is K-amenable;
- Any non-compact group having Kazhdan's property-(T) is not K-amenable.

Theorem (P. Julg and A. Valette, 1984)

Any group which acts properly on a tree is K-amenable.
In particular, free groups F_n are K-amenable.

K-amenability for groups acting on trees

We consider the simplest case $G = \mathbb{Z}$ acting on the 2-regular tree:

$$\cdots - - -3 - - -2 - - -1 - - 0 - - 1 - - 2 - - 3 - - \cdots$$

- $X_0 = \{ v_j \mid j \in \mathbb{Z} \}$: Vertices; $X_1 = \{ e_j \mid j \in \mathbb{Z}_{\neq 0} \}$: Edges;
- We have induced representations of G on $l^2(X_0)$ and $l^2(X_1)$;
- Fixing some vertex (say 0), we have a flow on the tree:

$$\cdots \leftarrow -3 \leftarrow -2 \leftarrow -1 \leftarrow 0 \rightarrow 1 \rightarrow 2 \rightarrow 3 \rightarrow \cdots$$

- This flow translates to the operator $\Gamma_0 : l^2(X_0) \to l^2(X_1)$
  $\Gamma_0(\delta_{v_0}) = 0, \ \Gamma(\delta_{v_n}) = (\delta_{e_n})$ for $n \in \mathbb{Z}_{\neq 0}$
- The cycle $[l^2(X_0) \xrightarrow{\Gamma_0} l^2(X_1)]$ defines an element in $R(G)$.

Theorem (P. Julg and A. Valette, 1984)

$[l^2(X_0) \xrightarrow{\Gamma_0} l^2(X_1)] = 1_G$ in $R(G)$.

Baum-Connes Conjecture for groups acting on trees

The story is not the end...

The Baum-Connes Conjecture (Baum and Connes, 1982)

The assembly map $\mu_G : K_*^G(EG) \to K_*(C_r^*(G))$ is an isomorphism of abelian groups for any group G.

- LH: Equivariant K-homology group of classifying space EG.
- RH: K-theory group of the reduced group C^*-algebra of G.

Theorem (Kasparov and Skandalis, 1991)

Any group which acts properly on a tree satisfy the Baum-Connes Conjecture, i.e. the assembly map μ_G is an isomorphism.

The proof follows the standard method called Dual-Dirac method. The method requires one to express 1_G in $R(G)$ in “a nice way”. The work by Julg and Valette contributes to the half of the job.

Further results

Theorem (Kasparov and Skandalis, 1991)

The assembly map μ_G is split injective for any group G which acts properly on a Euclidean building. In particular, the Novikov's Conjceture holds for such G.

Theorem (Brodzki, Guentner and Higson, 2016)

Any group which acts properly on a CAT(0)-cubical complex is K-amenable.

This work can be viewed as a higher-dimensional generalization of the work done by Julg and Valette.

Summary

- The K-amenability of groups was introduced to study K-theoritic amenability of groups.
- Julg and Valette found that the groups acting on trees provide nontrivial examples of K-amenable groups.
- There are higher dimensional analogues of what Julg and Valette did. Indeed, a group which acts properly on a CAT(0)-cubical complex is K-amenable.

THANK YOU VERY MUCH!!

References

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