The growth rates of ideal Coxeter polyhedra in hyperbolic 3-space

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Coxeter polyhedron in $\mathbb{H}^n$

$B^n := \{x \in \mathbb{R}^{n+1} \mid |x| < 1\}$

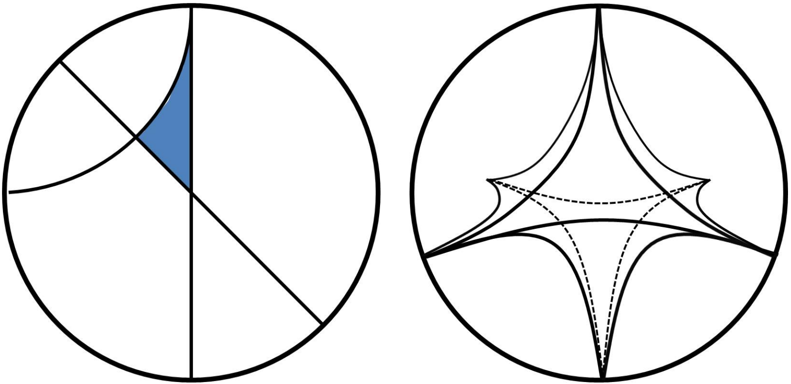
$\mathbb{H}^n := \left(B^n, \ \left(\frac{2}{1-|x|^2}\right)^2 \sum_{i=1}^n dx_i^2\right)$ **hyperbolic n-space**

Definition

A Coxeter polyhedron in $\mathbb{H}^n$ is an n-dimensional convex polyhedron in $\mathbb{H}^n$ with its dihedral angles are submultiples of π.

A convex polyhedron Δ in $\mathbb{H}^n$ is compact if $\overline{\Delta} \cap \partial\mathbb{H}^n$ is empty, and Δ is ideal if all of its vertices are in $\partial\mathbb{H}^n$.
If a convex polyhedron Δ with finite volume is non-compact, then it has vertices in $\partial\mathbb{H}^n$ finitely. We call these vertices cusps.

Coxeter polyhedron in $\mathbb{H}^n$



Examples of Coxeter polyhedra in hyperbolic spaces

Hyperbolic Coxeter group

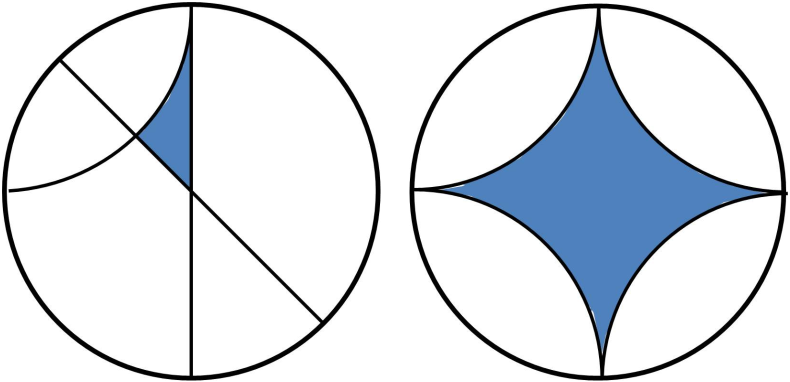
Definition

(G, S) is a Coxeter system if $G = \langle s \in S \mid (st)^{m_{s,t}} = 1 \ \mathit{for} \ s, t \in S \rangle$ with satisfying the following conditions

1. $m_{s,t} = m_{t,s}$
2. $\forall s \in S$, then $m_{s,s} = 1$
3. if $s \neq t$, then $m_{s,t} \in \{2, 3, \cdots, +\infty\}$

G is called Coxeter group.
A group generated by reflections obtained by each face of a Coxeter polyhedron (in $\mathbb{H}^n$) is a (n-dimensional hyperbolic) Coxeter group.

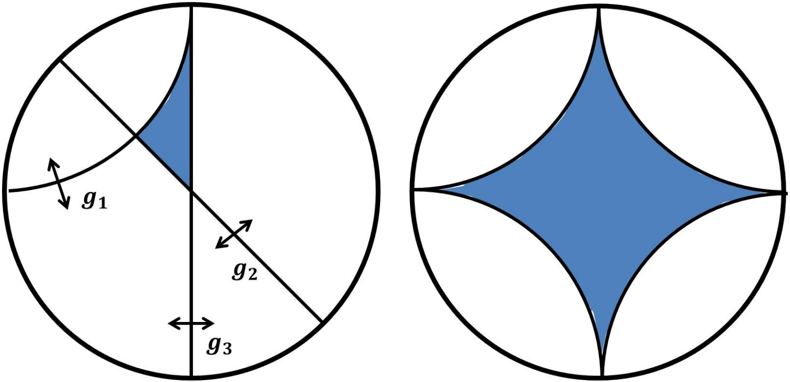
Hyperbolic Coxeter group



Coxeter polygons in $\mathbb{H}^2$

$$G_1 = \langle g_1, g_2, g_3 \mid g_1^2 = g_2^2 = g_3^2 = (g_1 g_2)^2 = (g_2 g_3)^4 = 1 \rangle$$
$$G_2 = \langle g_1, g_2, g_3, g_4 \mid g_1^2 = g_2^2 = g_3^2 = g_4^2 = 1 \rangle$$

Hyperbolic Coxeter group



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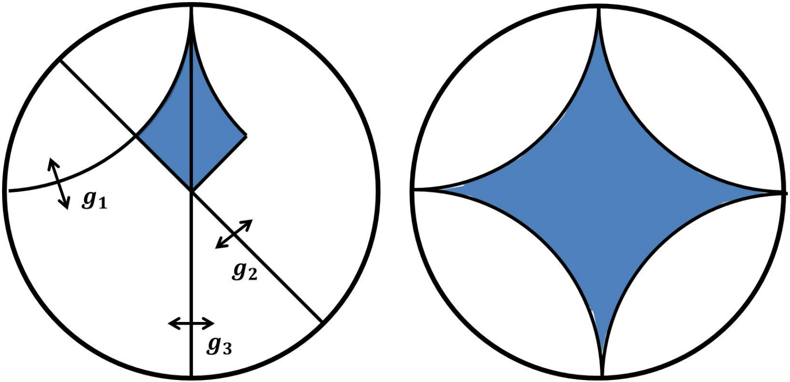
Hyperbolic Coxeter group



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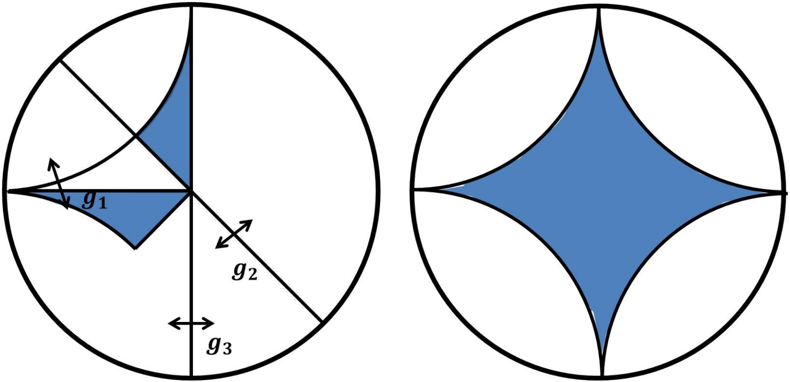

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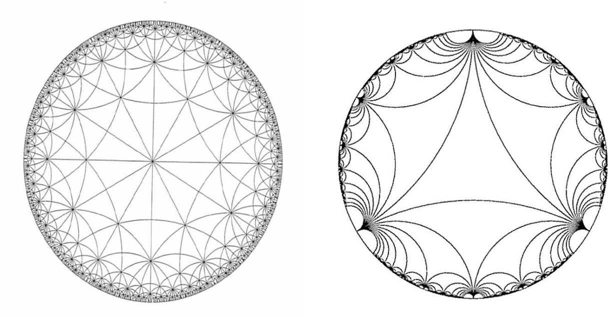
Hyperbolic Coxeter group



The tessellations of $\mathbb{H}^2$

Reference.
J. G. Ratcliffe, Foundations of Hyperbolic Manifolds Second Edditions, Graduate Texts in Math., **149**, Springer (2006)

Growth rate

G: finitely generated group with generating set S

$\ell_S(g) := \min\{n \in \mathbb{N} \mid g = s_1 \cdots s_n, s_i \in S\}$ $(\ell_S(1) = 0)$

$a_k := \#\{g \in G \mid \ell_S(g) = k\}$ $(\ a_0 = 1)$

Definition (Growth function of (G, S))

$$f_S(t) = \sum_{k=0}^{\infty} a_k t^k$$

Definition (Growth rate of (G, S))

$$\tau := \limsup_{k \to \infty} \sqrt[k]{a_k}$$

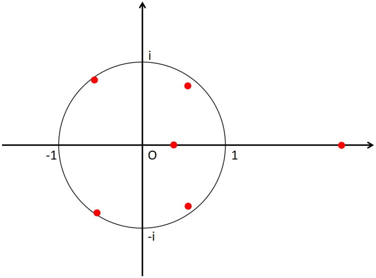
The radius of convergence of $f_S(t)$ is $\frac{1}{\tau}$

Previous works

Theorem (Cannon-Wagreich 1992, Parry 1993)

The growth rates of Coxeter groups obtained from compact Coxeter polyhedra in $\mathbb{H}^n$ for $n = 2, 3$ are Salem numbers.

A Salem number $\alpha > 1$ is a real algebraic integer if all its conjugates different from α are in the closed unit disk, and α has at least one conjugate on the unit circle.

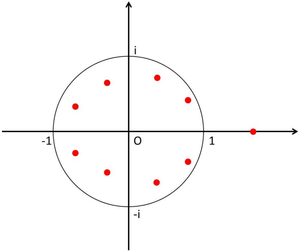


Previous works

Theorem (Floyd 1992)

The growth rates of Coxeter groups obtained from non-compact Coxeter polyhedra with finite area in $\mathbb{H}^2$ are Pisot numbers.

A Pisot number $\beta > 1$ is a real algebraic integer if all algebraic conjugates different from β are of absolute value < 1.

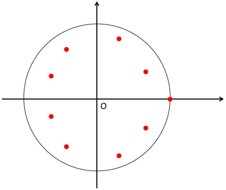


Previous works

Theorem (Kellerhals-Perren 2011)

The growth rates of Coxeter groups obtained from 4-dimensional compact hyperbolic Coxeter polyhedra with at most six 3-faces are Perron numbers.

A real algebraic integer $\gamma > 1$ is called Perron number if all its conjugates different from γ are of absolute value $< \gamma$.



Theorem (Komori-Umemoto 2012)

The growth rates of Coxeter groups obtained from 3-dimensional hyperbolic Coxeter polyhedra with four or five faces are Perron numbers.

Main result 1

Theorem (N-Kellerhals, Komori-Yukita)

The growth rate of an ideal Coxeter polyhedron in $\mathbb{H}^3$ is a Perron number.

Theorem (Solomon 1966)

$f_S(t)$: **the growth function of an irreducible spherical Coxeter group** (Γ, S) **is given by** $f_S(t) = \prod_{i=1}^{l}[m_i + 1]$
where $[n] := 1 + t + \cdots + t^{n-1}$ **and** $\{m_1 = 1, m_2, \cdots, m_l\}$ **is the set of exponents (as defined in [Humphreys]).**

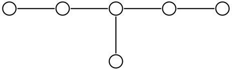
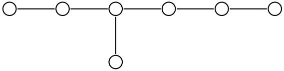
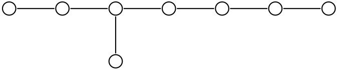
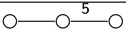
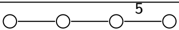
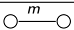
Symbols	Coxeter graphs	Exponents	growth functions
A_n		$1, 2, \cdots, n$	$[2, 3, \cdots, n+1]$
B_n		$1, 3, \cdots, 2n-1$	$[2, 4, \cdots, 2n]$
D_n		$1, 3, \cdots, 2n-3, n-1$	$[2, 4, \cdots, 2n-2][n]$
E_6		$1, 4, 5, 7, 8, 11$	$[2, 5, 6, 8, 9, 12]$
E_7		$1, 5, 7, 9, 11, 13, 17$	$[2, 6, 8, 10, 12, 14, 18]$
E_8		$1, 7, 11, 13, 17, 19, 23, 29$	$[2, 8, 12, 14, 18, 20, 24, 30]$
F_4		$1, 5, 7, 11$	$[2, 6, 8, 12]$
H_3		$1, 5, 9$	$[2, 6, 10]$
H_4		$1, 11, 19, 29$	$[2, 12, 20, 30]$
$I_2(m)$		$1, m-1$	$[2, m]$

Table: Exponents and growth functions of irreducible finite Coxeter groups [Kellerhals-Perren]

$[m, n] = [m][n]$

Theorem (Steinberg 1968)

$G = \langle S \mid R \rangle$: **Coxeter group**
$f_S(t)$: **the growth function of** G
$G_T = \langle T \mid R \rangle$: **Coxeter subgroup of** G **generated by** $T \subset S$
$f_T(t)$: **growth function of** G_T
$F = \{T \mid G_T$ **is finite**$\}$
Then

$$\frac{1}{f_S(t^{-1})} = \sum_{T \in F} \frac{(-1)^{|T|}}{f_T(t)}$$

P : **an ideal Coxeter polyhedron in** $\mathbb{H}^3$
$f_{\langle P \rangle}(t)$: **the growth function of** P

$$
\begin{aligned}
\frac{1}{f_{\langle P \rangle}(t^{-1})} &= 1 - \frac{f}{[2]} + \frac{e_2}{[2,2]} + \frac{e_3}{[2,3]} + \frac{e_4}{[2,4]} + \frac{e_6}{[2,6]} \\
&= 1 - \frac{f}{1+t} + \frac{6f - 2c - c_1 - 12}{2(1+t)^2} + \frac{2c - 2f + 2c_1 - c_2 + 4}{2(1+t)(1+t+t^2)} + \cdots \\
&\quad + \frac{2c - 2f - c_1 - c_2 + 4}{2(1+t)^2(1+t+t^2)(1-t+t^2)}
\end{aligned}
$$

$$f_{\langle P \rangle}(t) = \frac{[2,2,3](1+t^2)(1-t+t^2)}{(t-1)g_{\langle P \rangle}(t)}$$

$$
\begin{aligned}
g_{\langle P \rangle}(t) = {} & (c-1)t^7 + (c-f+1)t^6 + (c+f-c_1/2-4)t^5 \\
& + (2c - 2f + (c_1 - c_2)/2 + 2)t^4 + (2f - (c_1 - c_2)/2 - 6)t^3 \\
& + (c - f + c_1/2)t^2 + (f-3)t - 1
\end{aligned}
$$

Lemma (Komori-Umemoto 2012)

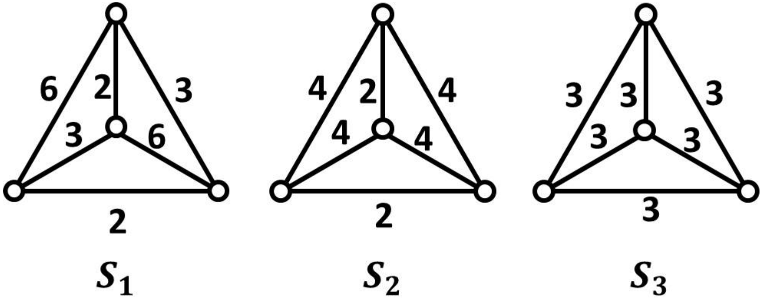
Consider the polynomial

$$g(t) = \sum_{k=1}^{n} b_k t^k - 1 \quad (n \geq 2),$$

where b_k is a non-negative integer.
Assume that the greatest common divisor of $\{k \in \mathbb{N} \mid b_k \neq 0\}$ is 1.
Then there is a real number r_0, $0 < r_0 < 1$ which is the unique zero of $g(t)$ having the smallest absolute value of all zeros of $g(t)$.

$$\begin{aligned} g_{\langle P \rangle}(t) = (c-1)t^7 + (c-f+1)t^6 + (c+f-c_1/2-4)t^5 \\ +(2c-2f+(c_1-c_2)/2+2)t^4 + (2f-(c_1-c_2)/2-6)t^3 \\ +(c-f+c_1/2)t^2 + (f-3)t - 1 \end{aligned}$$

Main result 2



Ideal Coxeter simplices in $\mathbb{H}^3$

$\tau(S_1) \sim 2.03074$
$vol(S_1) \sim 0.84579$

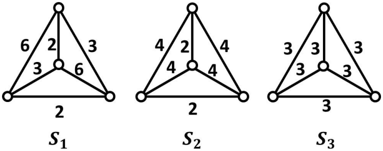
$\tau(S_2) \sim 2.13040$
$vol(S_2) \sim 0.91597$

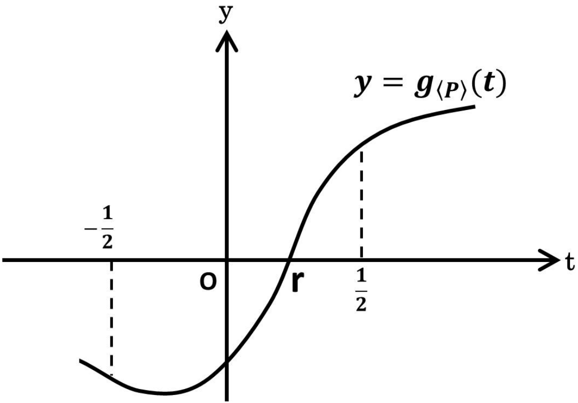
$\tau(S_3) \sim 2.30278$
$vol(S_3) \sim 1.01492$

Main result 2

Theorem (N-Kellerhals, Komori-Yukita)

The Coxeter tetrahedron S_1 has minimal growth rate among all ideal Coxeter polyhedra of finite volume in $\mathbb{H}^3$, and as such is unique. Its growth rate $\tau(S_1) \sim 2.03074$ is the Perron number with minimal polynomial $t^5 - t^4 - t^3 - t^2 - t - 3$.





P, P': **ideal Coxeter polyhedra in** $\mathbb{H}^3$
$\tau(P)$, $\tau(P')$: **the growth rates of** P **and** P'
If $g_{\langle P \rangle}(t) - g_{\langle P' \rangle}(t) > 0$ **for** $t \in (0, 1)$, **then** $\frac{1}{\tau(P)} < \frac{1}{\tau(P')}$.

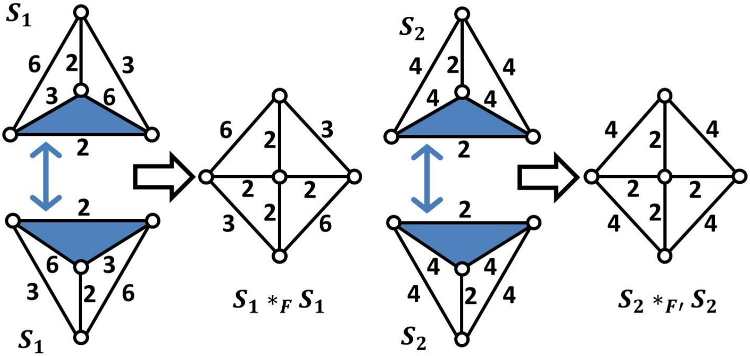
Main result 3

$\tau(\Delta)$: **the growth rate of an ideal Coxeter polyhedron Δ**

Theorem (N-Kellerhals)

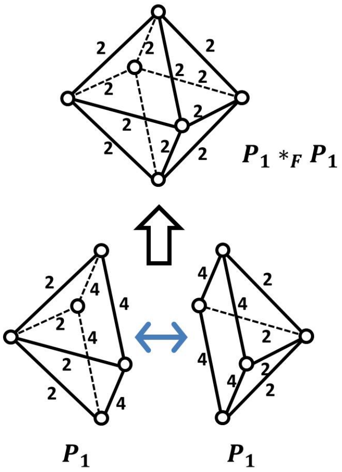
Let P and P' be ideal Coxeter polyhedra in $\mathbb{H}^3$. Suppose that P has a face F which is isometric to a face F' of P', and denote by $P *_F P'$ the ideal polyhedron arising by gluing P to P' along their isometric faces F and F'. If $P *_F P'$ is a Coxeter polyhedron, then

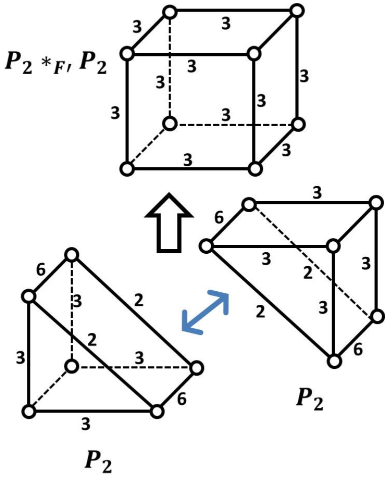
$$\tau(P *_F P') > \max\{\tau(P), \tau(P')\}.$$



$\tau(S_1) \sim 2.03074$
$\tau(S_1 *_F S_1) \sim 2.74738$

$\tau(S_2) \sim 2.13040$
$\tau(S_2 *_{F'} S_2) \sim 2.84547$





$\tau(P_1) \sim 2.84547$
$\tau(P_1 *_F P_1) = 5$

$\tau(P_2) \sim 3.16204$
$\tau(P_2 *_{F'} P_2) \sim 4.54138$

The growth rates of Coxeter polyhedra in $\mathbb{H}^3$

Theorem (Yukita)

The growth rate of a non-compact Coxeter polyhedron in $\mathbb{H}^3$ is a Perron number.

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Thank you for your attention!