A small normal generating set for the handlebody subgroup of the Torelli group

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(JSPS Research Fellowship for Young Scientists PD)

June 27, 2017
Boston University/Keio University Workshop 2017
Geometry and Mathematical Physics
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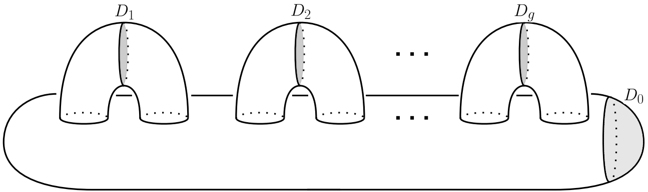
cf. arXiv:1607.06553

$H_g \subset S^3$: the oriented 3-dimensional handlebody of genus g.



$\Sigma_g := \partial H_g$, D_0: the disk on Σ_g,
$\Sigma_{g,1} := \Sigma_g - \mathrm{int} D_0$.

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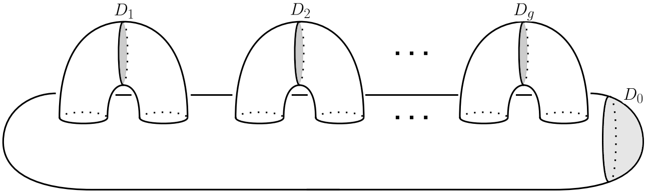

$\Sigma_g := \partial H_g$, D_0: the disk on Σ_g,
$\Sigma_{g,1} := \Sigma_g - \mathrm{int} D_0$.

$\mathrm{Diff}_+(\Sigma_{g,1}) := \{\varphi : \Sigma_g \to \Sigma_g$ ori.-pre. diffeo. $\mid \varphi|_{D_0} = \mathrm{id}_{D_0}\}$.

$\mathcal{M}_{g,1} := \mathrm{Diff}_+(\Sigma_{g,1})$/isotopy rel. D_0 : the *mapping class group* of $\Sigma_{g,1}$,
$\mathcal{H}_{g,1} := \{[\varphi] \in \mathcal{M}_{g,1} \mid \varphi$ extends to $H_g\}$: the *handlebody group*.

$\mathcal{M}_{g,1} \curvearrowright \mathrm{H}_1(\Sigma_g;\mathbb{Z})$
$\rightsquigarrow \Psi : \mathcal{M}_{g,1} \to \mathrm{Aut}\mathrm{H}_1(\Sigma_g;\mathbb{Z})$.

$\mathcal{I}_{g,1} := \ker \Psi$: the *Torelli group* of $\Sigma_{g,1}$,

$\mathcal{IH}_{g,1} := \ker\Psi|_{\mathcal{H}_{g,1}} = \mathcal{I}_{g,1} \cap \mathcal{H}_{g,1}$: the *handlebody subgroup of* $\mathcal{I}_{g,1}$.

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$\mathcal{K}_{g,1}$: the Johnson kernel of $\Sigma_{g,1}$.

Problem (Birman J. ('06), (corrected version))

$\cdots$. For these reasons it might be very useful to find generators for $\mathcal{IH}_{g,1}$ and/or $\mathcal{K}_{g,1} \cap \mathcal{H}_{g,1}$.

"these reasons" = a relationship with integral homology 3-spheres ($\mathbb{Z}\mathrm{H}S^3$s):

$$\lim_{g\to\infty} \mathcal{H}_{g,1} \backslash \mathcal{M}_{g,1} / - \mathcal{H}_{g,1} \xrightarrow{\cong} \{\text{oriented closed 3-mfd.s}\}$$

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Today's main result :

We obtain a generating set for $\mathcal{IH}_{g,1}$ when $g \geq 3$!!

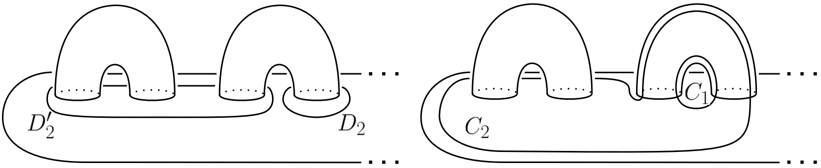
$\rightsquigarrow$ We answer Birman's problem for $\mathcal{IH}_{g,1}$ when $g \geq 3$.

Definition (Bounding pair (BP))

c_1, c_2: *s.c.c.s on* $\Sigma_{g,1}$,

- $\{c_1, c_2\}$: *a (genus-h) bounding pair ((genus-h) BP) on* $\Sigma_{g,1}$

$\overset{def}{\Longleftrightarrow} \begin{cases} c_1, c_2\text{: non-isotopic, non-separating in } \Sigma_{g,1}, \\ \exists \Sigma \approx \Sigma_{h,2}\text{: subsurface of } \Sigma_{g,1} \text{ s.t. } \partial\Sigma = c_1 \sqcup c_2. \end{cases}$



$\rightsquigarrow$ $\{D_2, D_2'\}$, $\{C_1, C_2\}$: genus-1 BPs.

- For a s.c.c. c on $\Sigma_{g,1}$,
  $t_c \in \mathcal{M}_{g,1}$: the *right-handed Dehn twist* along c.
- For a (genus-h) BP $\{c_1, c_2\}$,

$$t_{c_1} t_{c_2}^{-1} \in \mathcal{I}_{g,1}\text{: a } \textit{(genus-}h\textit{) BP-map along } \{c_1, c_2\}.$$

$\rightsquigarrow$ $t_{D_2} t_{D_2'}^{-1}$, $t_{C_1} t_{C_2}^{-1} \in \mathcal{I}_{g,1}$: genus-1 BP-maps.

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Theorem (Johnson ('79))

For $g \geq 3$, $\mathcal{I}_{g,1}$ is generated by genus-1 BP maps.

Theorem (Johnson ('83))

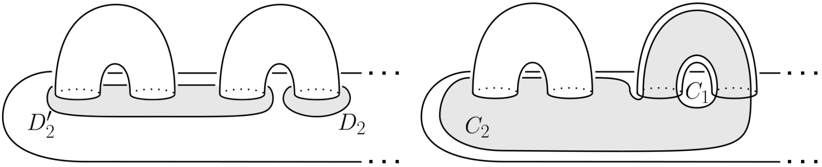
For $g \geq 3$, $\mathcal{I}_{g,1}$ is generated by finitely many BP maps.

Definition

$\{c_1, c_2\}$: *a genus-h BP on $\Sigma_{g,1}$,*

$\{c_1, c_2\}$: *a genus-h homotopical BP (genus-h HBP)*

$\overset{def}{\Longleftrightarrow}$ $\begin{cases} \text{each } c_i \ (i = 1, 2) \text{ does NOT bound a disk in } H_g, \\ \exists A\text{: annulus in } H_g \text{ s.t. } \partial A = c_1 \sqcup c_2. \end{cases}$



$\rightsquigarrow$ $\{C_1, C_2\}$: a genus-1 HBP.

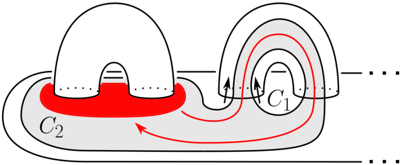
$\{c_1, c_2\}$: a genus-h HBP on $\Sigma_{g,1}$,

$$t_{c_1} t_{c_2}^{-1} \in \mathcal{I}_{g,1}\text{: a } \textit{genus-}h \textit{ HBP-map.}$$

$\rightsquigarrow$ $t_{C_1} t_{C_2}^{-1}$: a genus-1 HBP-map.

Remark

$\{c_1, c_2\}$: genus-h HBP $\Longrightarrow$ $t_{c_1} t_{c_2}^{-1} \in \mathcal{IH}_{g,1}$.



Definition

G: a group, H: a normal subgroup of G, $x_1, x_2, \ldots, x_n \in H$,
H is normally generated by $x_1, x_2, \ldots, x_n$ in G

$$\overset{\text{def}}{\Longleftrightarrow} H = \left\langle \{ g x_i g^{-1} \mid g \in G,\ 1 \leq i \leq n \} \right\rangle.$$

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Example

- (Mumford ('67))
  $\mathcal{M}_{g,1}$ is normally generated by t_c (c: non-sep.) in $\mathcal{M}_{g,1}$.
- (Johnson ('79))
  For $g \geq 3$, $\mathcal{I}_{g,1}$ is normally generated by a genus-1 BP-map in $\mathcal{M}_{g,1}$.

Theorem (O.)

For $g \geq 3$, $\mathcal{IH}_{g,1}$ is normally generated by $t_{C_1} t_{C_2}^{-1}$ in $\mathcal{H}_{g,1}$.
In particular, $\mathcal{IH}_{g,1}$ is generated by genus-1 HBP-maps.

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- A genus-1 HBP-map is not always conjugate to $t_{C_1} t_{C_2}^{-1}$ in $\mathcal{H}_{g,1}$.

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$\rightsquigarrow$

- We give a necessary and sufficient condition that a genus-1 HBP-map is conjugate to $t_{C_1}t_{C_2}^{-1}$ in $\mathcal{H}_{g,1}$.
- We give examples of genus-1 HBP-maps which are NOT conjugate to $t_{C_1}t_{C_2}^{-1}$ in $\mathcal{H}_{g,1}$.

Outline of the proof of the main theorem

$* \in \partial D_0 \subset \Sigma_g = \partial H_g$, $\mathcal{H}_{g,1} \curvearrowright \pi_1(H_g, *) \cong F_g$.
$\rightsquigarrow$ We have the homomorphism $\eta : \mathcal{H}_{g,1} \to \mathrm{Aut}F_g$.

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We can check $\eta(\mathcal{IH}_{g,1}) = IA_g$, where $IA_g := \ker(\mathrm{Aut} F_g \to GL(g, \mathbb{Z}))$.
$\rightsquigarrow$ we have the exact sequence

$$1 \longrightarrow \ker \eta|_{\mathcal{IH}_{g,1}} \longrightarrow \mathcal{IH}_{g,1} \overset{\eta|_{\mathcal{IH}_{g,1}}}{\longrightarrow} \mathrm{IA}_g \longrightarrow 1.$$

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- Magnus ('35) gave an explicit finite generating set $\{C_i\}$ for IA_g when $g \geq 1$.
- Pitsch ('09) gave an infinite generating set $\{D_j\}$ for $\ker \eta|_{\mathcal{IH}_{g,1}}$ when $g \geq 3$.

Lifts of C_i's $\cdots$ conjugations of $t_{C_1} t_{C_2}^{-1}$ in $\mathcal{H}_{g,1}$.
$\rightsquigarrow$ We show that Pitsch's generators $\{D_j\}$ are products of conjugations of $t_{C_1} t_{C_2}^{-1}$ in $\mathcal{H}_{g,1}$!!

Problem

Theorem (Johnson ('79) (again))

For $g \geq 3$, $\mathcal{I}_{g,1}$ is a normally generated by a genus-1 BP map in $\mathcal{M}_{g,1}$.

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Problem

Is $\mathcal{IH}_{g,1}$ finitely generated for $g \geq 3$?

Thank you for your attention!!