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Truncation of the Yamabe invariant

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§1. The Yamabe invariant and its sign

M^n a $\underbrace{\text{closed}}$ n-dim mfd

- compact without boundary
- not necessarily connected or oriented

scalar curvature

$$\operatorname{sign}(M) := \begin{cases} +1 & \text{if } M \text{ admits a Riem. met. } g \text{ with } R_g > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$Y(M) := \sup_{\mathcal{C}} \inf_{g \in \mathcal{C}} \frac{\int_M R_g \, d\mu_g}{\left(\int_M d\mu_g\right)^{\frac{n-2}{n}}} \quad \text{(Yamabe invariant)}$$

Rmk

- $Y(M) \in (-\infty, \underbrace{n(n-1)\omega_n^{2/n}}_{= Y(S^n),\ \omega_n = \mathrm{Vol}^n(S^n(1))}]$
- $Y(M) > 0 \iff \operatorname{sign}(M) = +1$, i.e.,

$$\begin{array}{ccc} & \{M \mid \text{closed } n\text{-dim}\} & \\ {\scriptstyle \max\{Y(\cdot),0\}} \swarrow & \circlearrowleft & \searrow {\scriptstyle \operatorname{sign}} \\ \mathbb{R} & \longrightarrow & \mathbb{R} \\ a & \longmapsto & \frac{a}{|a|} \end{array}$$

- The Kazdan-Warner trichotomy (for connected M) does not correspond to the sign of $Y(M)$.

§2. Bordism invariance

Recall:

- $M^n \underset{\text{of codim } n-k}{\xrightarrow{\text{surgery}}} N^n := (M^n - S^k \times \mathring{D}^{n-k}) \cup \bar{D}^{k+1} \times S^{n-k-1}$

  $\text{embedding} \uparrow$

  $S^k \times \bar{D}^{n-k}$

- $M^n \underset{\text{bordant}}{\sim} N^n$ (i.e. $\exists W^{n+1}$ s.t. $\partial W^{n+1} = M \sqcup N$)

  $\Leftrightarrow M^n \xrightarrow{\text{surg}} \cdots \xrightarrow{\text{surg}} N^n$

- $S^n \underset{\text{bordant}}{\sim} T^n$, $\text{sign}(S^n) = 1 \neq 0 = \text{sign}(T^n)$

Thm (Ammann – Dahl – Humbert) (enlargeability obstruction or minimal hypersurface techniques)

$\forall n \;\exists \Lambda_n > 0$ s.t.

$$M^n \underset{\text{codim} \geq 3}{\xrightarrow{\text{surg of}}} N^n \Rightarrow Y(N) \geq \min\{Y(M), \Lambda_n\}.$$

Rmk This refines:

- $\text{sign}(N) \geq \text{sign}(M)$ (Gromov – Lawson, Schoen – Yau)
- $Y(N) \geq Y(M)$ for codim $= n$ (Kobayashi
- $Y(N) \geq Y(M)$ if $Y(M) \leq 0$ (Petean – Yun

The surgery theorems allow one to draw

$$\begin{tikzcd}
\{M^n \mid \text{closed spin}, \pi_1(M) = \Gamma\} \arrow[r] \arrow[dr, "Y_{tr}"] \arrow[ddr, "\text{sign}"'] & \Omega^{Spin}_n(B\Gamma) \arrow[d, "ADH"] \\
& \mathbb{R} \arrow[d, dashed] \\
& \mathbb{R}
\end{tikzcd}$$

$\circlearrowleft$ GL

where Γ finitely presented, $n \geq 5$, $Y_{tr}(M)$ ~~$:=$~~ $= \min\{\max\{Y(M), 0\}, \Lambda_n\}$

§3. Result

Let $\Omega_n^{Spin}(B\Gamma) \xrightarrow{ABS} ko_n(B\Gamma)$ be the homomorphism of abelian groups associated with the Atiyah–Bott–Shapiro orientation $MSpin \to ko$.

<u>Thm</u> (Ammann – Joachim – O.). The map $\Omega_n^{Spin}(B\Gamma) \xrightarrow{ADH} \mathbb{R}$ further factors through $ko_n(B\Gamma)$, possibly after taking $\Lambda_n > 0$ smaller.

$$\begin{tikzcd}
\Omega_n^{Spin}(B\Gamma) \arrow[r, "ABS"] \arrow[d, "ADH"'] & ko_n(B\Gamma) \arrow[dl, "?"', "\text{Thm}"] \arrow[ddl, "?", "\text{Stolz, Jung, Führing}"'] \\
\mathbb{R} \arrow[d] & \\
\mathbb{R} &
\end{tikzcd}$$

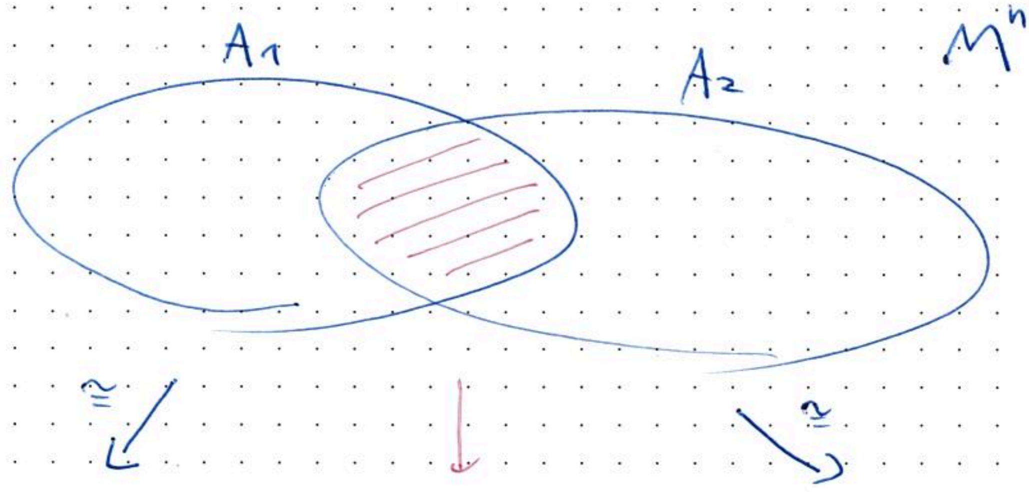
after $\otimes \mathbb{Z}_{(2)}$ after $\otimes \mathbb{Z}[\frac{1}{2}]$

Stolz, Jung, and Führing showed that $\ker(ABS)$ consists of mfds with $\mathrm{sign} = +1$.

We have to show: $\exists\, C_n > 0$ s.t.

$$[M, f] \in \ker(ABS),\ f_* : \pi_1(M) \xrightarrow{\cong} \Gamma$$
$$\Longrightarrow Y(M) \geq C_n.$$

This amounts to estimating the Yamabe invariant for mfds of the following type:



$B_1 \times \mathbb{H}P^2$ $\quad B_{12} \times \mathbb{H}P^2 \times \mathbb{H}P^2$ $\quad B_2 \times \mathbb{H}P^2$

Step 1: Define a one-parameter family of metrics $\{g_t\}$ on M by expanding B_1, B_2, and B_{12}.

Step 2: Show that g_t converges to either $\mathbb{R}^{n-8} \times \mathbb{H}P^2$ or $\mathbb{R}^{n-16} \times \mathbb{H}P^2 \times \mathbb{H}P^2$ in the pointed Gromov–Hausdorff topology.

Step 3:
- continuity of the Yamabe constant (c.f. Akutagawa–Florit–Petean, Honda)
- Petean, Petean–Ruiz,