Random Holonomy and Algebraic Structures

Steve Rosenberg
(with Michael Murray and Raymond Vozzo)

Boston University, U. of Adelaide

June 30, 2017

Overview

QFT/QM computes the time evolution of a system by Schrödinger operators

$$e^{it\Delta} : \mathcal{H} \to \mathcal{H}$$

for some operator Δ on a Hilbert space $\mathcal{H}$.

Overview

QFT/QM computes the time evolution of a system by Schrödinger operators

$$e^{it\Delta} : \mathcal{H} \to \mathcal{H}$$

for some operator Δ on a Hilbert space $\mathcal{H}$.

Formally, for $f \in \mathcal{H}$, we have

$$\langle e^{it\Delta} f, g \rangle = \int_{\mathcal{P}_t} e^{itL(\gamma)} d\mu(\gamma),$$

where $\mathcal{P}_t$ is the space of paths starting at f and ending at g at time t, and $d\mu(\gamma)$ is some measure.

Overview

QFT/QM computes the time evolution of a system by Schrödinger operators

$$e^{it\Delta} : \mathcal{H} \to \mathcal{H}$$

for some operator Δ on a Hilbert space $\mathcal{H}$.

Formally, for $f \in \mathcal{H}$, we have

$$\langle e^{it\Delta} f, g \rangle = \int_{\mathcal{P}_t} e^{itL(\gamma)} d\mu(\gamma),$$

where $\mathcal{P}_t$ is the space of paths starting at f and ending at g at time t, and $d\mu(\gamma)$ is some measure.

Problem: $d\mu$ probably doesn't exist. Even if $d\mu$ exists as some Wiener measure, the integral doesn't exist in general.

Overview

QFT/QM computes the time evolution of a system by Schrödinger operators

$$e^{it\Delta} : \mathcal{H} \to \mathcal{H}$$

for some operator Δ on a Hilbert space $\mathcal{H}$.

Formally, for $f \in \mathcal{H}$, we have

$$\langle e^{it\Delta} f, g \rangle = \int_{\mathcal{P}_t} e^{itL(\gamma)} d\mu(\gamma),$$

where $\mathcal{P}_t$ is the space of paths starting at f and ending at g at time t, and $d\mu(\gamma)$ is some measure.

Problem: $d\mu$ probably doesn't exist. Even if $d\mu$ exists as some Wiener measure, the integral doesn't exist in general.

Partial solution: TQFT extracts meaningful information from formal path integrals. We treat the mysterious path integral as a functor Z.

Overview

QFT/QM computes the time evolution of a system by Schrödinger operators

$$e^{it\Delta} : \mathcal{H} \to \mathcal{H}$$

for some operator Δ on a Hilbert space $\mathcal{H}$.

Formally, for $f \in \mathcal{H}$, we have

$$\langle e^{it\Delta} f, g \rangle = \int_{\mathcal{P}_t} e^{itL(\gamma)} d\mu(\gamma),$$

where $\mathcal{P}_t$ is the space of paths starting at f and ending at g at time t, and $d\mu(\gamma)$ is some measure.

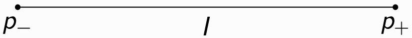
Problem: $d\mu$ probably doesn't exist. Even if $d\mu$ exists as some Wiener measure, the integral doesn't exist in general.

Partial solution: TQFT extracts meaningful information from formal path integrals. We treat the mysterious path integral as a functor Z.

In this talk, we'll try to put back some of the analysis into the TQFT picture. We choose a particular Z, and find algebraic structures in the theory.

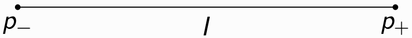
1d TQFT

Let $\mathcal{C}$ be the category whose objects are 0-dimensional oriented manifolds and whose morphisms are oriented bordisms I between points.



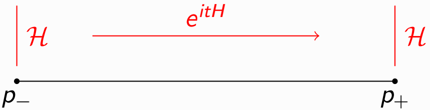
1d TQFT

Let $\mathcal{C}$ be the category whose objects are 0-dimensional oriented manifolds and whose morphisms are oriented bordisms I between points.



Let $\mathcal{V}$ be the category of vector spaces and homomorphisms.

Define the functor $Z : \mathcal{C} \to \mathcal{V}$ by: $Z(p_+) = \mathcal{H}, Z(p_-) = \mathcal{H}^*$. Set $Z(I) = e^{itH}$, whatever that is, so $Z(I) \in \mathrm{Hom}(\mathcal{H}, \mathcal{H}) = \mathcal{H}^* \otimes \mathcal{H}$.



$$Z(I) \in Z(p_-) \otimes Z(p_+) := Z(\{p_-\} \cup \{p_+\}) = Z(\partial I)$$

2d TQFT

A 2d TQFT is a functor Z which assigns to an oriented 1d closed manifold $kS^1 = \sqcup_{i=1}^{k} S^1$ a vector space $Z(kS^1) = V_k$ and to each oriented 2d manifold with boundary (Σ, kS^1) a vector $Z(\Sigma) \in V_k = Z(\partial\Sigma)$. The axioms

$$Z(S^1_-) = Z(S^1_+)^*, \; Z\left(S^1 \sqcup S^1\right) = Z(S^1) \otimes Z(S^1)$$

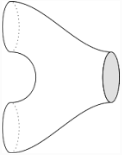
imply that pairs of pants give rise to products and coproducts:

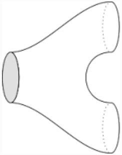
2d TQFT

A 2d TQFT is a functor Z which assigns to an oriented 1d closed manifold $kS^1 = \sqcup_{i=1}^{k} S^1$ a vector space $Z(kS^1) = V_k$ and to each oriented 2d manifold with boundary (Σ, kS^1) a vector $Z(\Sigma) \in V_k = Z(\partial\Sigma)$. The axioms

$$Z(S^1_-) = Z(S^1_+)^*, \; Z\left(S^1 \sqcup S^1\right) = Z(S^1) \otimes Z(S^1)$$

imply that pairs of pants give rise to products and coproducts:

M_1 

 M_2

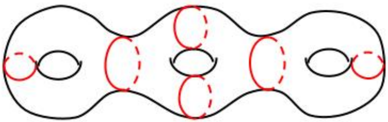
gives

$$Z(M_1) \in V^* \otimes V^* \otimes V = Hom(V \otimes V, V)$$

$$Z(M_2) \in V^* \otimes V \otimes V = Hom(V, V \otimes V)$$

String theory and pairs of pants

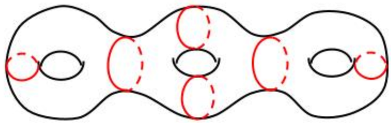
String theory: The main object is $\mathrm{Maps}(\Sigma^2, M)$. A map $f : \Sigma^2 \to M$ is thought of as a circle/string moving inside M:

$f :$  $\longrightarrow M$

$\overrightarrow{t}$

String theory and pairs of pants

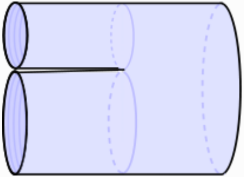
String theory: The main object is $\mathrm{Maps}(\Sigma^2, M)$. A map $f : \Sigma^2 \to M$ is thought of as a circle/string moving inside M:

$f :$  $\longrightarrow M$

$\overrightarrow{t}$

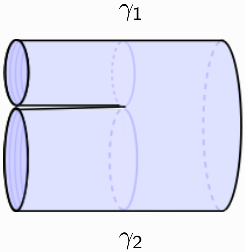
A map decomposes into maps of pairs of pants into M:

γ_1

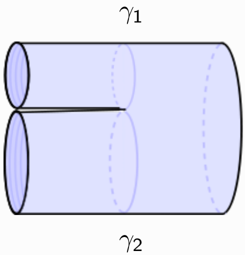


γ_2

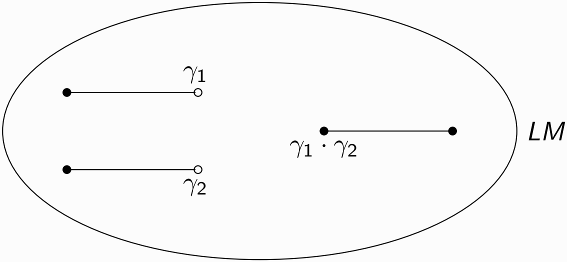
Pairs of pants in loop space



Pairs of pants in loop space



A pair of pants in the loop space $LM = \mathrm{Maps}(S^1, M)$ looks like:



Choosing Z: Holonomy as a (failed) 2d TQFT

Start with a complex Hermitian bundle (E, ∇) with connection over LM.

Example: Given a complex Hermitian bundle (E', ∇') with connection over M, we get a "loopified" infinite rank bundle $E = LE' \to LM$, with fiber $LE'_\gamma = \{$the sections of E' above $\gamma\}$. There is an associated L^2 connection on LE.

Example: If $E' = T_{\mathbb{C}}M$, then $LE' = T_{\mathbb{C}}LM$.

Choosing Z: Holonomy as a (failed) 2d TQFT

Start with a complex Hermitian bundle (E, ∇) with connection over LM.

Example: Given a complex Hermitian bundle (E', ∇') with connection over M, we get a "loopified" infinite rank bundle $E = LE' \to LM$, with fiber $LE'_\gamma = \{$the sections of E' above $\gamma\}$. There is an associated L^2 connection on LE.

Example: If $E' = T_{\mathbb{C}}M$, then $LE' = T_{\mathbb{C}}LM$.

For $Z(kS^1)$, choose k fixed loops in M, i.e. k points $x_1, \ldots, x_k$ in LM. Set $Z(x_i) = E_{x_i}$. This is already not a TQFT.

Choosing Z: Holonomy as a (failed) 2d TQFT

Start with a complex Hermitian bundle (E, ∇) with connection over LM.

Example: Given a complex Hermitian bundle (E', ∇') with connection over M, we get a "loopified" infinite rank bundle $E = LE' \to LM$, with fiber $LE'_\gamma = \{$the sections of E' above $\gamma\}$. There is an associated L^2 connection on LE.

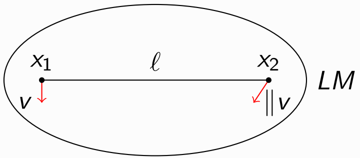
Example: If $E' = T_{\mathbb{C}}M$, then $LE' = T_{\mathbb{C}}LM$.

For $Z(kS^1)$, choose k fixed loops in M, i.e. k points $x_1, \ldots, x_k$ in LM. Set $Z(x_i) = E_{x_i}$. This is already not a TQFT.

For $k = 2$, define

$$Z(\text{bordism between } k = 2 \text{ points})$$

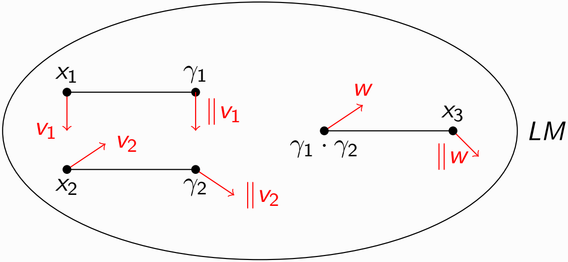
by choosing a curve ℓ in LM between the points, i.e. a cylinder between loops x_1, x_2 in M. Set $Z(\ell) \in \mathrm{Hom}(E_{x_1}, E_{x_2})$ to be the holonomy along the cylinder.



Holonomy as a 2d TQFT II

Assume E is an LM-groupoid.

For $k = 3$, Z(pair of pants) should be a product $E_{x_1} \otimes E_{x_2} \to E_{x_3}$.

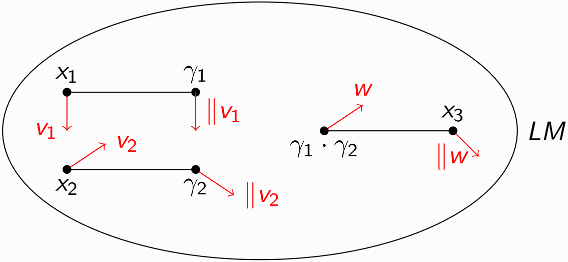


Here $w = \gamma_1 \cdot \| v_2 + \| v_1 \cdot \gamma_2$. So $Z(\text{pair of pants})(v_1, v_2) = \| w = v_3$.

Holonomy as a 2d TQFT II

Assume E is an LM-groupoid.

For $k = 3$, Z(pair of pants) should be a product $E_{x_1} \otimes E_{x_2} \to E_{x_3}$.



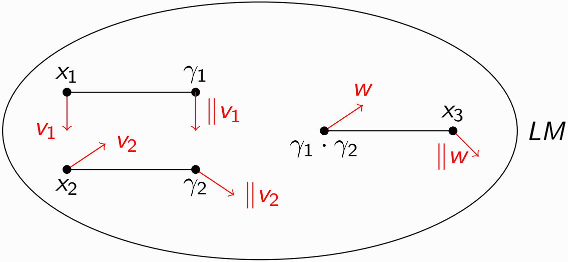
Here $w = \gamma_1 \cdot \| v_2 + \| v_1 \cdot \gamma_2$. So $Z(\text{pair of pants})(v_1, v_2) = \| w = v_3$.

Reading left to right, the product takes $E_{x_1} \otimes E_{x_2}$ to E_{x_3}. Reading right to left, we see a coproduct.

Holonomy as a 2d TQFT II

Assume E is an LM-groupoid.

For $k = 3$, Z(pair of pants) should be a product $E_{x_1} \otimes E_{x_2} \to E_{x_3}$.

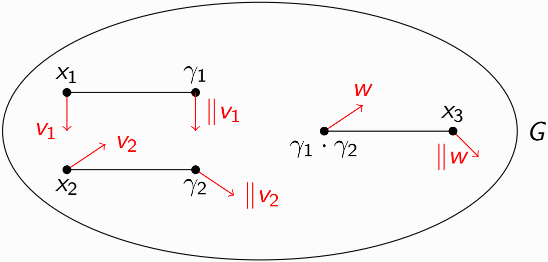


Here $w = \gamma_1 \cdot \| v_2 + \| v_1 \cdot \gamma_2$. So $Z(\text{pair of pants})(v_1, v_2) = \| w = v_3$.

Reading left to right, the product takes $E_{x_1} \otimes E_{x_2}$ to E_{x_3}. Reading right to left, we see a coproduct.

Problem: In string theory, we want to integrate over/average over all pairs of pants. What is the measure?

Holonomy on a Lie group



This picture is fine if G is a Lie group and (E, ∇) is a G-bundle.

The "broken tuning fork" is no longer a bordism. We're moving away from TQFT formalism.

We want to average over $\{\gamma_1, \gamma_2\}$ and over all paths. How to do this?

Average holonomy on a Riemannian manifold M

Let $W_{t,x}$ be Wiener (probability) measure on the path space

$$P_{t,x} = \{\gamma : [0,t] \to M : \gamma(0) = x, \gamma \text{ continuous}\}.$$

For $A \subset P_{t,x}$, $W_{t,x}(A) = \int_{P_{t,x}} \chi_A \, dW_{t,x} = \int_A dW_{t,x}$ is the probability $\mathbb{P}[\gamma \in A]$ that a random path lies in A.

Similarly, there is a pinned Wiener measure W_{t,x_1,x_2} on the path space

$$P_{t,x_1,x_2} = \{\gamma : [0,t] \to M : \gamma(0) = x_1, \gamma(t) = x_2, \gamma \text{ continuous}\}.$$

Average holonomy on a Riemannian manifold M

Let $W_{t,x}$ be Wiener (probability) measure on the path space

$$P_{t,x} = \{\gamma : [0,t] \to M : \gamma(0) = x, \gamma \text{ continuous}\}.$$

For $A \subset P_{t,x}$, $W_{t,x}(A) = \int_{P_{t,x}} \chi_A \, dW_{t,x} = \int_A dW_{t,x}$ is the probability $\mathbb{P}[\gamma \in A]$ that a random path lies in A.

Similarly, there is a pinned Wiener measure W_{t,x_1,x_2} on the path space

$$P_{t,x_1,x_2} = \{\gamma : [0,t] \to M : \gamma(0) = x_1, \gamma(t) = x_2, \gamma \text{ continuous}\}.$$

Example: For $U \subset M$, and for $A = \{\gamma : \gamma(t) \in U\}$,

$$\mathbb{P}[\gamma \in A] = \mathbb{P}[\gamma(t) \in U] = W_{t,x}(A) = \int_U K_t(x,y) \mathrm{dvol}(y),$$

where $K_t(x,y)$ is the heat kernel on M.

Average holonomy on a Riemannian manifold M

Let $W_{t,x}$ be Wiener (probability) measure on the path space

$$P_{t,x} = \{\gamma : [0,t] \to M : \gamma(0) = x, \gamma \text{ continuous}\}.$$

For $A \subset P_{t,x}$, $W_{t,x}(A) = \int_{P_{t,x}} \chi_A \, dW_{t,x} = \int_A dW_{t,x}$ is the probability $\mathbb{P}[\gamma \in A]$ that a random path lies in A.

Similarly, there is a pinned Wiener measure W_{t,x_1,x_2} on the path space

$$P_{t,x_1,x_2} = \{\gamma : [0,t] \to M : \gamma(0) = x_1, \gamma(t) = x_2, \gamma \text{ continuous}\}.$$

Example: For $U \subset M$, and for $A = \{\gamma : \gamma(t) \in U\}$,

$$\mathbb{P}[\gamma \in A] = \mathbb{P}[\gamma(t) \in U] = W_{t,x}(A) = \int_U K_t(x,y)\mathrm{dvol}(y),$$

where $K_t(x,y)$ is the heat kernel on M.

Here the time t heat flow of an initial temperature distribution $f \in C^\infty(M)$ is given by

$$(e^{-t\Delta}f)(x) = \int_M K_t(x,y)f(y)\mathrm{dvol}(y).$$

Intuition for the Wiener measure

The heat kernel looks like

$$K_t(x,y) = e^{-d^2(x,y)/4t}(c_0 t^{-\dim(M)/2} + c_1 t^{-(\dim(M)/2)+1} + \ldots).$$

So for small t, $K_t(x,y)$ is very large if $x = y$, and very small if $x \neq y$.

Intuition for the Wiener measure

The heat kernel looks like

$$K_t(x,y) = e^{-d^2(x,y)/4t}(c_0 t^{-\dim(M)/2} + c_1 t^{-(\dim(M)/2)+1} + \ldots).$$

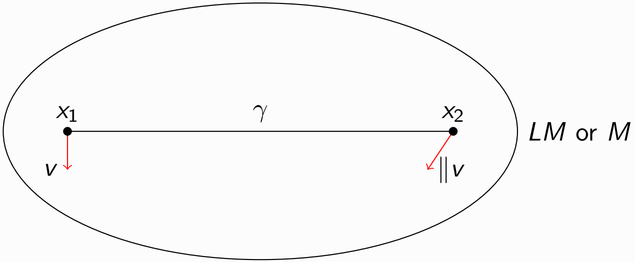
So for small t, $K_t(x,y)$ is very large if $x = y$, and very small if $x \neq y$.

$$\mathbb{P}[\gamma(t) \in U] = \int_U K_t(x,y)\mathrm{dvol}(y)$$

For a path starting at x, $K_t(x,y)$ is small if $x \notin U$ and t is small: there is a low probability of traveling from x to U in a short time, and this probability is even lower if $d(x,U)$ is big.

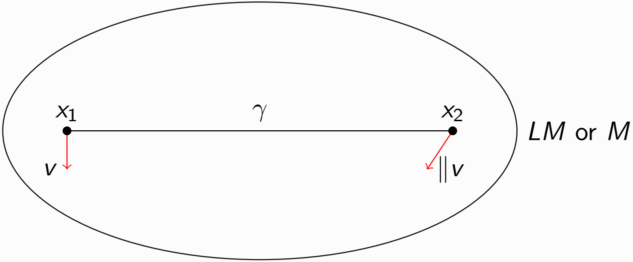
Average holonomy on a Riemannian manifold II

As a warm-up, consider the $k = 2$ case. We want the average holonomy over paths from x_1 to x_2



Average holonomy on a Riemannian manifold II

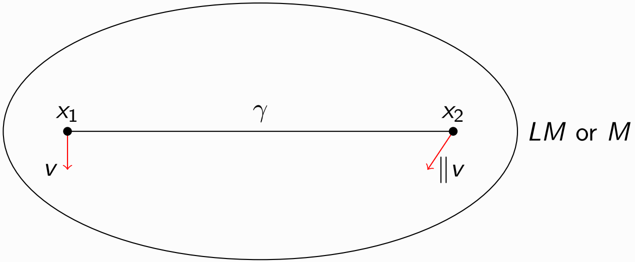
As a warm-up, consider the $k = 2$ case. We want the average holonomy over paths from x_1 to x_2



The average holonomy is $\int_{P_{t,x_1,x_2}} \|_\gamma v \; dW_{t,x_1,x_2}(\gamma)$, for the case of M.

Average holonomy on a Riemannian manifold II

As a warm-up, consider the $k = 2$ case. We want the average holonomy over paths from x_1 to x_2



The average holonomy is $\int_{P_{t,x_1,x_2}} \|_\gamma v \; dW_{t,x_1,x_2}(\gamma)$, for the case of M.

Note: 1) It is unknown if LM has a Wiener measure in general. LG might be ok.
2) Parallel transport exists along continuous paths by solving a stochastic ODE.

Average holonomy for pairs of pants: disintegration

Recall the Fubini theorem for a rectangle:

$$\int_{[a,b]\times[c,d]} f(x,y)\frac{dx}{b-a}\frac{dy}{d-c} = \int_{[a,b]}\left(\int_{[c,d]} f(x,y)\frac{dy}{d-c}\right)\left(\frac{dx}{(b-a)}\right)$$

$$= \int_{[a,b]}\left(\int_{[c,d]} f(x,y)\frac{dy}{d-c}\right)\pi_*\left(\frac{dx\ dy}{(b-a)(d-c)}\right),$$

where $\pi : (x,y) \mapsto x$ and $\pi_*(...) = dx/(b-a)$. The support of each $dy/(d-c)$ is in $\pi^{-1}(x)$.

Average holonomy for pairs of pants: disintegration

Recall the Fubini theorem for a rectangle:

$$\int_{[a,b]\times[c,d]} f(x,y)\frac{dx}{b-a}\frac{dy}{d-c} = \int_{[a,b]}\left(\int_{[c,d]} f(x,y)\frac{dy}{d-c}\right)\left(\frac{dx}{(b-a)}\right)$$

$$= \int_{[a,b]}\left(\int_{[c,d]} f(x,y)\frac{dy}{d-c}\right)\pi_*\left(\frac{dx\ dy}{(b-a)(d-c)}\right),$$

where $\pi : (x,y) \mapsto x$ and $\pi_*(\ldots) = dx/(b-a)$. The support of each $dy/(d-c)$ is in $\pi^{-1}(x)$.

This is the basic example of a disintegration formula: For $\pi : (Y,\mu) \to (X, \pi_*\mu)$ and $f : Y \to \mathbb{R}$ a measurable map on a probability space, in many cases there are probability measures μ_x supported in $\pi^{-1}(x)$ such that

$$\int_Y f(y)d\mu(y) = \int_X \left(\int_{\pi^{-1}(x)} f(y)d\mu_x(y)\right)\pi_*\mu(x).$$

Disintegration and Wiener measures

$$\pi : (Y, \mu) \to (X, \pi_* \mu), \ \int_Y f(y) d\mu(y) = \int_X \left(\int_{\pi^{-1}(x)} f(y) d\mu_x(y) \right) \pi_* \mu(x).$$

Example: $(Y, \mu) = (P_{t,x}, W_{t,x})$, $X = M$, $\pi = \mathrm{ev}_t : Y \to M$, where $\mathrm{ev}_t(\gamma) = \gamma(t)$.

Then $(\mathrm{ev}_{t,*} W_{t,x})(y) = K_t(x, y) \mathrm{dvol}(y)$, $\mathrm{ev}_t^{-1}(y) = \{\gamma : \gamma(t) = y\} = P_{t,x,y}$. For $f : P_{t,x} \to \mathbb{R}$, the average value of f is

$$\mathbb{E}[f] = \int_{P_{t,x}} f \ dW_{t,x} = \int_M \left(\int_{P_{t,x,y}} f \ dW_{t,x,y} \right) K_t(x, y) \ dy.$$

Disintegration and Wiener measures

$$\pi : (Y, \mu) \to (X, \pi_* \mu), \ \int_Y f(y) d\mu(y) = \int_X \left(\int_{\pi^{-1}(x)} f(y) d\mu_x(y) \right) \pi_* \mu(x).$$

Example: $(Y, \mu) = (P_{t,x}, W_{t,x})$, $X = M$, $\pi = \mathrm{ev}_t : Y \to M$, where $\mathrm{ev}_t(\gamma) = \gamma(t)$.

Then $(\mathrm{ev}_{t,*} W_{t,x})(y) = K_t(x,y)\mathrm{dvol}(y)$, $\mathrm{ev}_t^{-1}(y) = \{\gamma : \gamma(t) = y\} = P_{t,x,y}$. For $f : P_{t,x} \to \mathbb{R}$, the average value of f is

$$\mathbb{E}[f] = \int_{P_{t,x}} f \ dW_{t,x} = \int_M \left(\int_{P_{t,x,y}} f \ dW_{t,x,y} \right) K_t(x,y) \ dy.$$

Example: $Y = P_{t,x,y}, X = M, \pi = \mathrm{ev}_s$ for $s \in (0,t)$ some intermediate time.
Then $\mathrm{ev}_s^{-1}(z) = \{\gamma : \gamma(s) = z\} = P_{s,x,z} \times P_{t-s,z,y}$ and

$$\int_{P_{t,x,y}} f \ dW_{t,x,y} = \int_M \left(\int_{P_{s,x,z} \times P_{t-s,z,y}} f \ d(W_{s,x,z} \times W_{t-s,z,y}) \right) \mathrm{ev}_{s,*} W_{t,x,y}.$$

Disintegration and Wiener measures

$$\pi : (Y, \mu) \to (X, \pi_* \mu), \ \int_Y f(y) d\mu(y) = \int_X \left(\int_{\pi^{-1}(x)} f(y) d\mu_x(y) \right) \pi_* \mu(x).$$

Example: $(Y, \mu) = (P_{t,x}, W_{t,x})$, $X = M$, $\pi = \mathrm{ev}_t : Y \to M$, where $\mathrm{ev}_t(\gamma) = \gamma(t)$.

Then $(\mathrm{ev}_{t,*} W_{t,x})(y) = K_t(x, y) \mathrm{dvol}(y)$, $\mathrm{ev}_t^{-1}(y) = \{\gamma : \gamma(t) = y\} = P_{t,x,y}$. For $f : P_{t,x} \to \mathbb{R}$, the average value of f is

$$\mathbb{E}[f] = \int_{P_{t,x}} f \ dW_{t,x} = \int_M \left(\int_{P_{t,x,y}} f \ dW_{t,x,y} \right) K_t(x, y) \ dy.$$

Example: $Y = P_{t,x,y}, X = M, \pi = \mathrm{ev}_s$ for $s \in (0, t)$ some intermediate time. Then $\mathrm{ev}_s^{-1}(z) = \{\gamma : \gamma(s) = z\} = P_{s,x,z} \times P_{t-s,z,y}$ and

$$\int_{P_{t,x,y}} f \ dW_{t,x,y} = \int_M \left(\int_{P_{s,x,z} \times P_{t-s,z,y}} f \ d(W_{s,x,z} \times W_{t-s,z,y}) \right) \mathrm{ev}_{s,*} W_{t,x,y}.$$

All these measures can be expressed in terms of the heat kernel.

Average holonomy and disintegration

From the average holonomy formula $\int_{P_{t,x,y}} \|_\gamma v \; dW_{t,x,y}(\gamma)$ and the disintegration formula

$$\int_{P_{t,x,y}} f \; dW_{t,x,y} = \int_M \left(\int_{P_{s,x,z} \times P_{t-s,z,y}} f \; d(W_{s,x,z} \times W_{t-s,z,y}) \right) \mathrm{ev}_{s,*} W_{t,x,y},$$

we get a holonomy disintegration formula

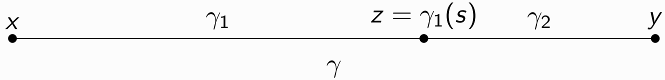
Average holonomy and disintegration

From the average holonomy formula $\int_{P_{t,x,y}} \|_\gamma v \; dW_{t,x,y}(\gamma)$ and the disintegration formula

$$\int_{P_{t,x,y}} f \; dW_{t,x,y} = \int_M \left(\int_{P_{s,x,z} \times P_{t-s,z,y}} f \; d(W_{s,x,z} \times W_{t-s,z,y}) \right) \mathrm{ev}_{s,*} W_{t,x,y},$$

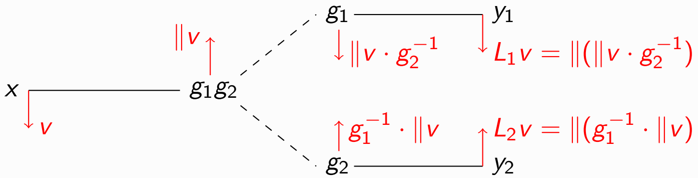
we get a holonomy disintegration formula

$$\int_{P_{t,x,y}} \|_{t,x \to y}^{\gamma} v \; dW_{t,x,y}(\gamma)$$
$$= \int_M (\mathrm{ev}_{s,*} W_{t,x,y})(z) \left(\int_{P_{t-s,z,y}} dW_{t-s,z,y} \|_{t-s,z,y}^{\gamma_2} \left(\int_{P_{s,x,z}} dW_{s,x,z} \|_{s,x \to z}^{\gamma_1} v \right) \right).$$



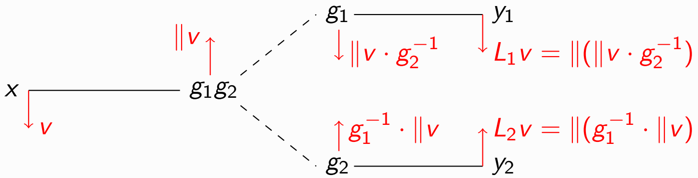
Average holonomy and coproducts

For a pair of pants "going right" in G:



Average holonomy and coproducts

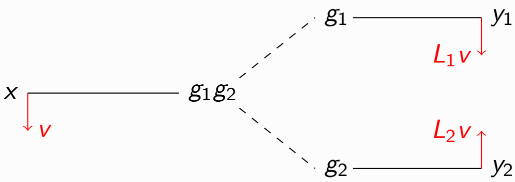
For a pair of pants "going right" in G:



the average holonomy of $v \in E_x$ is $\Delta v \in E_{y_1} \otimes E_{y_2}$ given by $\Delta v = L_1(v) \otimes 1 + 1 \otimes L_2(v)$, with

$$L_1(v) = \int_{\substack{(g_1, g_2) \\ \in G \times G}} \mathrm{ev}_{s,*} W_{t,x,y_1 g_2}(g_1)\, \mathrm{ev}_{s,*} W_{t,x,g_1 y_2}(g_2) \left(\int_{P_{t-s,g_1,y_1}} dW_{t-s,g_1,y_1} \Big\| \left(\left[\int_{P_{s,x,g_1 g_2}} \|v(x) dW_{s,x,g_1 g_2} \right] \cdot g_2^{-1} \right) \right)$$

Average holonomy and coproducts II



$$L_2(v) =$$

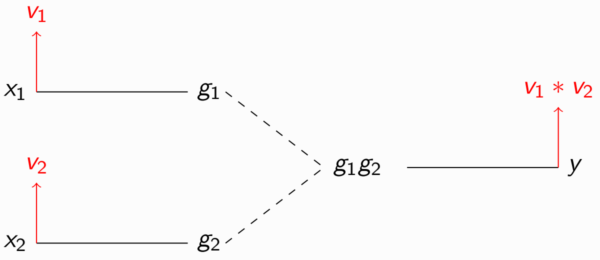
$$\int_{\substack{(g_1,g_2)\\ \in G\times G}} \mathrm{ev}_{s,*} W_{t,x,y_1 g_2}(g_1)\ \mathrm{ev}_{s,*} W_{t,x,g_1 y_2}(g_2)$$

$$\left(\int_{P_{t-s,g_2,y_2}} dW_{t-s,g_2,y_2} \Big\| \left(g_1^{-1} \cdot \left[\int_{P_{s,x,g_1 g_2}} \| v(x) dW_{s,x,g_1 g_2}\right]\right)\right)$$

Note that $\Delta v = L_1(v) \otimes 1 + 1 \otimes L_2(v)$ means that $\Delta v \in \mathcal{T}E_{y_1} \otimes \mathcal{T}E_{y_2}$. Extending by $\Delta(v_1 \otimes v_2) = \Delta(v_1) \otimes \Delta(v_2)$ gives $\Delta : \mathcal{T}E_x \to \mathcal{T}E_{y_1} \otimes \mathcal{T}E_{y_2}$.

Average holonomy and products

The product $* : \mathcal{T}E_{x_1} \otimes \mathcal{T}E_{x_2} \to \mathcal{T}E_y$ is given similarly:



$$
\begin{aligned}
v_1 * v_2 &= v_1 *_{s,t} v_2 \\
&= \int_{\substack{(g_1,g_2)\\ \in G\times G}} \mathrm{ev}_{s,*} W_{t,x_1,yg_2^{-1}}(g_1) \mathrm{ev}_{s,*} W_{t,x_2,g_1^{-1}y}(g_2) \\
&\quad \int_{P_{t-s,g_1g_2,y}} \| \left[\left(\int_{P_{s,x_1,g_1}} \| v_1 \; dW_{s,x_1,g_1} \right) \cdot g_2 \right] dW_{t-s,g_1g_2,y} \\
&\quad \otimes (\text{similar with } g_1 \cdot)
\end{aligned}
$$

Time for examples

Example: $G = \mathbb{C}^n, (E, \nabla) = (T\mathbb{C}^n, d)$.

$\|^\gamma v = v$ for all γ. Thus $L_1 = L_2 = \mathrm{Id}$, and the product on $\mathcal{T}\mathbb{C}^n$ is the usual

$$(v_1 \otimes \ldots \otimes v_k) * (w_1 \otimes \ldots \otimes w_\ell) = v_1 \otimes \ldots \otimes v_k \otimes w_1 \otimes \ldots \otimes w_\ell.$$

$\Delta v = v \otimes 1 + 1 \otimes v$ induces the standard shuffle product on $\mathcal{T}\mathbb{C}^n$:

$$\Delta(v_1 \otimes \ldots \otimes v_n) = \sum_{p=0}^{n} \sum_{Sh_{p,n-p}} (v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(p)}) \otimes (v_{\sigma(p+1)} \otimes \ldots \otimes v_{\sigma(n)})$$

Time for examples

Example: $G = \mathbb{C}^n, (E, \nabla) = (T\mathbb{C}^n, d)$.

$\|^\gamma v = v$ for all γ. Thus $L_1 = L_2 = \mathrm{Id}$, and the product on $\mathcal{T}\mathbb{C}^n$ is the usual

$$(v_1 \otimes \ldots \otimes v_k) * (w_1 \otimes \ldots \otimes w_\ell) = v_1 \otimes \ldots \otimes v_k \otimes w_1 \otimes \ldots \otimes w_\ell.$$

$\Delta v = v \otimes 1 + 1 \otimes v$ induces the standard shuffle product on $\mathcal{T}\mathbb{C}^n$:

$$\Delta(v_1 \otimes \ldots \otimes v_n) = \sum_{p=0}^{n} \sum_{Sh_{p,n-p}} (v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(p)}) \otimes (v_{\sigma(p+1)} \otimes \ldots \otimes v_{\sigma(n)})$$

The product/coproduct form a Hopf algebra: there is a compatibility condition between the associative product and coassociative coproduct, there are units and counts, there is an antipode $v_1 \otimes \ldots v_k \mapsto (-1)^k v_1 \otimes \ldots v_k$.

Time for examples

Example: $G = \mathbb{C}^n, (E, \nabla) = (T\mathbb{C}^n, d)$.

$\|^\gamma v = v$ for all γ. Thus $L_1 = L_2 = \mathrm{Id}$, and the product on $\mathcal{T}\mathbb{C}^n$ is the usual

$$(v_1 \otimes \ldots \otimes v_k) * (w_1 \otimes \ldots \otimes w_\ell) = v_1 \otimes \ldots \otimes v_k \otimes w_1 \otimes \ldots \otimes w_\ell.$$

$\Delta v = v \otimes 1 + 1 \otimes v$ induces the standard shuffle product on $\mathcal{T}\mathbb{C}^n$:

$$\Delta(v_1 \otimes \ldots \otimes v_n) = \sum_{p=0}^{n} \sum_{Sh_{p,n-p}} \left(v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(p)}\right) \otimes \left(v_{\sigma(p+1)} \otimes \ldots \otimes v_{\sigma(n)}\right)$$

The product/coproduct form a Hopf algebra: there is a compatibility condition between the associative product and coassociative coproduct, there are units and counts, there is an antipode $v_1 \otimes \ldots v_k \mapsto (-1)^k v_1 \otimes \ldots v_k$.

A classic example of a Hopf algebra: $H^*(G)$ with product given by $\cup$, and the multiplication $m : G \times G \to G$ induces the coproduct

$$m^* : H^*(G) \to H^*(G \times G) \simeq H^*(G) \otimes H^*(G).$$

More examples

Example: $G = S^1$, $(E, \nabla) = (S^1 \times \mathbb{C}, d)$.
Again, we get the tensor Hopf algebra. $v_1 * v_2 = v_1 \otimes v_2$.

More examples

Example: $G = S^1$, $(E, \nabla) = (S^1 \times \mathbb{C}, d)$.
Again, we get the tensor Hopf algebra. $v_1 * v_2 = v_1 \otimes v_2$.

Example: $G = S^1$, $(E, \nabla) = (S^1 \times \mathbb{C}, d_\psi)$, where d_ψ is the flat $U(1)$-connection with holonomy $e^{i\psi}$.

More examples

Example: $G = S^1$, $(E, \nabla) = (S^1 \times \mathbb{C}, d)$.
Again, we get the tensor Hopf algebra. $v_1 * v_2 = v_1 \otimes v_2$.

Example: $G = S^1$, $(E, \nabla) = (S^1 \times \mathbb{C}, d_\psi)$, where d_ψ is the flat $U(1)$-connection with holonomy $e^{i\psi}$.

Set $x_1 = x_2 = y = 1 \in S^1$. Let

$$\mu^k_{s,e^{i\theta_1}} = W_{s,1,e^{i\theta_1}}(\{\text{paths that wrap } k \text{ times around } S^1\}).$$

More examples

Example: $G = S^1$, $(E, \nabla) = (S^1 \times \mathbb{C}, d)$.
Again, we get the tensor Hopf algebra. $v_1 * v_2 = v_1 \otimes v_2$.

Example: $G = S^1$, $(E, \nabla) = (S^1 \times \mathbb{C}, d_\psi)$, where d_ψ is the flat $U(1)$-connection with holonomy $e^{i\psi}$.

Set $x_1 = x_2 = y = 1 \in S^1$. Let

$$\mu^k_{s,e^{i\theta_1}} = W_{s,1,e^{i\theta_1}}(\{\text{paths that wrap } k \text{ times around } S^1\}).$$

Then

$$v_1 * v_2 = \frac{1}{Z} \int_{(e^{i\theta_1}, e^{i\theta_2}) \in T^2} d\theta_1 d\theta_2 \; K_s(1, e^{i\theta_1}) K_{t-s}(e^{i\theta_2}, e^{i\theta_1}) K_t^{-1}(1, e^{i\theta_1})$$

$$K_s(1, e^{i\theta_2}) K_{t-s}(e^{i\theta_1}, e^{i\theta_2}) K_t^{-1}(1, e^{i\theta_2})$$

$$\sum_\ell e^{i\psi(\ell + (\theta_1+\theta_2)/2\pi)} \mu^\ell_{t-s, e^{i\theta_1} e^{i\theta_2}} \left(\sum_k e^{i\psi(k+\theta_1/2\pi)} \mu^k_{s,e^{i\theta_1}} v_1 \right.$$

$$\left. \otimes \sum_k e^{i\psi(k+\theta_2/2\pi)} \mu^k_{s,e^{i\theta_2}} v_2 \right).$$

Deformations of the Hopf algebra

The product

$$v_1 * v_2 = \frac{1}{Z}\int_{(e^{i\theta_1},e^{i\theta_2})\in T^2} d\theta_1 d\theta_2\ K_s(1,e^{i\theta_1})K_{t-s}(e^{i\theta_2},e^{i\theta_1})K_t^{-1}(1,e^{i\theta_1})$$

$$K_s(1,e^{i\theta_2})K_{t-s}(e^{i\theta_1},e^{i\theta_2})K_t^{-1}(1,e^{i\theta_2})$$

$$\sum_\ell e^{i\psi(\ell+(\theta_1+\theta_2)/2\pi)}\mu^\ell_{t-s,e^{i\theta_1}e^{i\theta_2}}\left(\sum_k e^{i\psi(k+\theta_1/2\pi)}\mu^k_{s,e^{i\theta_1}}v_1\right.$$

$$\left.\otimes\sum_k e^{i\psi(k+\theta_2/2\pi)}\mu^k_{s,e^{i\theta_2}}v_2\right),$$

with

$$Z = \text{the red stuff},$$

Deformations of the Hopf algebra

The product

$$v_1 * v_2 = \frac{1}{Z} \int_{(e^{i\theta_1}, e^{i\theta_2}) \in T^2} d\theta_1 d\theta_2 \; K_s(1, e^{i\theta_1}) K_{t-s}(e^{i\theta_2}, e^{i\theta_1}) K_t^{-1}(1, e^{i\theta_1})$$

$$K_s(1, e^{i\theta_2}) K_{t-s}(e^{i\theta_1}, e^{i\theta_2}) K_t^{-1}(1, e^{i\theta_2})$$

$$\sum_\ell e^{i\psi(\ell + (\theta_1+\theta_2)/2\pi)} \mu^\ell_{t-s, e^{i\theta_1} e^{i\theta_2}} \left(\sum_k e^{i\psi(k+\theta_1/2\pi)} \mu^k_{s, e^{i\theta_1}} v_1 \right.$$

$$\left. \otimes \sum_k e^{i\psi(k+\theta_2/2\pi)} \mu^k_{s, e^{i\theta_2}} v_2 \right),$$

with

$$Z = \text{the red stuff},$$

is determined by the deformation of the flat connection on $E = S^1 \times \mathbb{C} \to S^1$ to the flat connection with holonomy $e^{i\psi}$. So we've deformed the Hopf tensor algebra to some product/coproduct structures, which are parametrized by the space of S^1-connections, i.e. in general by $\mathfrak{g} \times GL(\mathbb{C}^n)$.

Deformations of the Hopf algebra

The product

$$v_1 * v_2 = \frac{1}{Z}\int_{(e^{i\theta_1},e^{i\theta_2})\in T^2} d\theta_1 d\theta_2 \; K_s(1,e^{i\theta_1})K_{t-s}(e^{i\theta_2},e^{i\theta_1})K_t^{-1}(1,e^{i\theta_1})$$

$$K_s(1,e^{i\theta_2})K_{t-s}(e^{i\theta_1},e^{i\theta_2})K_t^{-1}(1,e^{i\theta_2})$$

$$\sum_{\ell} e^{i\psi(\ell+(\theta_1+\theta_2)/2\pi)}\mu^{\ell}_{t-s,e^{i\theta_1}e^{i\theta_2}} \left(\sum_k e^{i\psi(k+\theta_1/2\pi)}\mu^k_{s,e^{i\theta_1}} v_1 \right.$$

$$\left. \otimes \sum_k e^{i\psi(k+\theta_2/2\pi)}\mu^k_{s,e^{i\theta_2}} v_2 \right),$$

with

$$Z = \text{the red stuff},$$

is determined by the deformation of the flat connection on $E = S^1 \times \mathbb{C} \to S^1$ to the flat connection with holonomy $e^{i\psi}$. So we've deformed the Hopf tensor algebra to some product/coproduct structures, which are parametrized by the space of S^1-connections, i.e. in general by $\mathfrak{g} \times GL(\mathbb{C}^n)$.

Question: What is the algebraic structure on the deformed product/coproduct?

Dealing with nonassociativity

From now on, set $x = y_1 = y_2$, so the product/coproduct takes $\mathcal{T}E_x$ to $\mathcal{T}E_x$.

Dealing with nonassociativity

From now on, set $x = y_1 = y_2$, so the product/coproduct takes $\mathcal{T}E_x$ to $\mathcal{T}E_x$.

Is the deformed product associative? No.

$$(a * b) * c = L_1^2 a \otimes L_1 L_2 b \otimes L_2 c, \;\; a * (b * c) = L_1 a \otimes L_2 L_1 b \otimes L_2^2 c.$$

Dealing with nonassociativity

From now on, set $x = y_1 = y_2$, so the product/coproduct takes $\mathcal{T}E_x$ to $\mathcal{T}E_x$.

Is the deformed product associative? No.

$$(a * b) * c = L_1^2 a \otimes L_1 L_2 b \otimes L_2 c, \;\; a * (b * c) = L_1 a \otimes L_2 L_1 b \otimes L_2^2 c.$$

Is the deformed coproduct coassociative? No.

Dealing with nonassociativity

From now on, set $x = y_1 = y_2$, so the product/coproduct takes $\mathcal{T}E_x$ to $\mathcal{T}E_x$.

Is the deformed product associative? No.

$$(a * b) * c = L_1^2 a \otimes L_1 L_2 b \otimes L_2 c, \;\; a * (b * c) = L_1 a \otimes L_2 L_1 b \otimes L_2^2 c.$$

Is the deformed coproduct coassociative? No.

Are the product and coproduct compatible? No.

Measuring nonassociativity

"Definition": An A_∞-algebra is a differential graded algebra A in which associativity fails only "up to homotopy."

Measuring nonassociativity

"Definition": An A_∞-algebra is a differential graded algebra A in which associativity fails only "up to homotopy."

Example: Take a fiber E_x of E with the $*$-product on $\mathcal{T}E_x$ as the algebra. Then

$$g_3(a \otimes b \otimes c) = (a * b) * c - a * (b * c)$$

measures nonassociativity.

$$g_3 \in \mathrm{Hom}(\mathcal{T}E^{\otimes 3}, \mathcal{T}E).$$

Measuring nonassociativity

"Definition": An A_∞-algebra is a differential graded algebra A in which associativity fails only "up to homotopy."

Example: Take a fiber E_x of E with the $*$-product on $\mathcal{T}E_x$ as the algebra. Then

$$g_3(a \otimes b \otimes c) = (a * b) * c - a * (b * c)$$

measures nonassociativity.

$$g_3 \in \mathrm{Hom}(\mathcal{T}E^{\otimes 3}, \mathcal{T}E).$$

If $\mathcal{T}E$ is graded and has a differential d (?!), then d induces a differential ∂ on $\mathrm{Hom}^*(\mathcal{T}E^{\otimes 3}, \mathcal{T}E)$. If $g_3 = \partial m_3$ for some $m_3 \in \mathrm{Hom}^{-1}(\mathcal{T}E^{\otimes 3}, \mathcal{T}E)$, then the $*$-product induces an associative product on $(H^*(\mathrm{Hom}(....), \partial)$.

Measuring nonassociativity

"Definition": An A_∞-algebra is a differential graded algebra A in which associativity fails only "up to homotopy."

Example: Take a fiber E_x of E with the $*$-product on $\mathcal{T}E_x$ as the algebra. Then

$$g_3(a \otimes b \otimes c) = (a * b) * c - a * (b * c)$$

measures nonassociativity.

$$g_3 \in \mathrm{Hom}(\mathcal{T}E^{\otimes 3}, \mathcal{T}E).$$

If $\mathcal{T}E$ is graded and has a differential d (?!), then d induces a differential ∂ on $\mathrm{Hom}^*(\mathcal{T}E^{\otimes 3}, \mathcal{T}E)$. If $g_3 = \partial m_3$ for some $m_3 \in \mathrm{Hom}^{-1}(\mathcal{T}E^{\otimes 3}, \mathcal{T}E)$, then the $*$-product induces an associative product on $(H^*(\mathrm{Hom}(....), \partial)$.

Similarly, we want to keep track of all products $a * (b * (c * r)), (a * b) * (c * r), \ldots$ in an explicit expression $g_4(a \otimes b \otimes c \otimes r)$ and solve $\partial m_4 = g_4$. etc.

A_∞-algebras

An A_∞-algebra is a differential graded algebra with $g_k \in \mathrm{Hom}^{1-k}(A^{\otimes k}, A)$, $k \in \mathbb{Z}^+$, measuring failure of associativity of k-fold products, and operations $m_k \in \mathrm{Hom}^{2-k}(A^{\otimes k}, A)$ with $\partial m_k = g_k$.

A_∞-algebras

An A_∞-algebra is a differential graded algebra with $g_k \in \mathrm{Hom}^{1-k}(A^{\otimes k}, A)$, $k \in \mathbb{Z}^+$, measuring failure of associativity of k-fold products, and operations $m_k \in \mathrm{Hom}^{2-k}(A^{\otimes k}, A)$ with $\partial m_k = g_k$.

Examples: The singular chain complex for the based loop space ΩM. The Fukaya A_∞-category on the symplectic side of mirror symmetry.

A_∞-algebras

An A_∞-algebra is a differential graded algebra with $g_k \in \mathrm{Hom}^{1-k}(A^{\otimes k}, A)$, $k \in \mathbb{Z}^+$, measuring failure of associativity of k-fold products, and operations $m_k \in \mathrm{Hom}^{2-k}(A^{\otimes k}, A)$ with $\partial m_k = g_k$.

Examples: The singular chain complex for the based loop space ΩM. The Fukaya A_∞-category on the symplectic side of mirror symmetry.

We need a differential on $\mathcal{T}E$. As a trick, we replace $\mathcal{T}E$ with

$$A = \Lambda^* E^* \otimes \mathcal{T}E,$$

the Fock space of E.

$$(\alpha \otimes v) * (\beta \otimes w) = (\alpha \wedge \beta) \otimes (v * w)$$

A_∞-algebras

An A_∞-algebra is a differential graded algebra with $g_k \in \mathrm{Hom}^{1-k}(A^{\otimes k}, A)$, $k \in \mathbb{Z}^+$, measuring failure of associativity of k-fold products, and operations $m_k \in \mathrm{Hom}^{2-k}(A^{\otimes k}, A)$ with $\partial m_k = g_k$.

Examples: The singular chain complex for the based loop space ΩM. The Fukaya A_∞-category on the symplectic side of mirror symmetry.

We need a differential on $\mathcal{T}E$. As a trick, we replace $\mathcal{T}E$ with

$$A = \Lambda^* E^* \otimes \mathcal{T}E,$$

the Fock space of E.

$$(\alpha \otimes v) * (\beta \otimes w) = (\alpha \wedge \beta) \otimes (v * w)$$

This has a differential on the exterior algebra part: take a nonzero vector $v \in E^*$ and set

$$d_v(v_1 \wedge ... \wedge v_k \otimes (a_1 \otimes \ldots \otimes a_r)) = v \wedge v_1 \wedge ... \wedge v_k \otimes (a_1 \otimes \ldots \otimes a_r)).$$

The A_∞-algebra in our context

We want to find $m_k \in \mathrm{Hom}(A^{\otimes k}, A)$ with $\partial m_k = g_k$. We certainly need $\partial g_k = 0$.

The A_∞-algebra in our context

We want to find $m_k \in \mathrm{Hom}(A^{\otimes k}, A)$ with $\partial m_k = g_k$. We certainly need $\partial g_k = 0$.

Examples: (1) $m_3(a \otimes b \otimes c) = \imath_{v^\sharp} La \otimes L^2 b \otimes L^2 c - (-1)^{|a|+|b|} L^2 a \otimes L^2 b \otimes \imath_{v^\sharp} Lc$.
(Here $L = L_1$ or L_2.)

The A_∞-algebra in our context

We want to find $m_k \in \mathrm{Hom}(A^{\otimes k}, A)$ with $\partial m_k = g_k$. We certainly need $\partial g_k = 0$.

Examples: (1) $m_3(a \otimes b \otimes c) = \imath_{v\sharp} La \otimes L^2 b \otimes L^2 c - (-1)^{|a|+|b|} L^2 a \otimes L^2 b \otimes \imath_{v\sharp} Lc$.
(Here $L = L_1$ or L_2.)

(2)

$$m_4(a \otimes b \otimes c \otimes d)$$
$$= -(-1)^{|c|} L^3 a \otimes L^3 b \otimes \imath_{v\sharp} L^2 c \otimes \imath_{v\sharp} Ld + (-1)^{|a|+|b|+|c|} \imath_{v\sharp} La \otimes L^3 b \otimes L^3 c \otimes \imath_{v\sharp} L^2 d$$
$$+ (-1)^{|a|+|b|+|c|} \imath_{v\sharp} L^2 a \otimes L^3 b \otimes L^3 c \otimes \imath_{v\sharp} Ld - (-1)^{|a|} \imath_{v\sharp} La \otimes \imath_{v\sharp} L^2 b \otimes L^3 c \otimes L^3 d$$
$$- (-1)^{|a|+|b|} \imath_{v\sharp} L^2 a \otimes L^2 b \otimes \imath_{v\sharp} L^2 c \otimes L^2 d - (-1)^{|b|+|c|} L^2 a \otimes \imath_{v\sharp} L^2 b \otimes L^2 c \otimes \imath_{v\sharp} L^2 d.$$

Main algebraic result

Theorem

If $\partial g_k = 0$*, then there exists an explicit solution to the equation* $\partial m_k = g_k$*. Thus if* $\partial g_k = 0$ *for all* k*,*

$$A = \Lambda^* E \otimes \mathcal{T} E$$

is an A_∞*-algebra.*

Main algebraic result

Theorem

If $\partial g_k = 0$, then there exists an explicit solution to the equation $\partial m_k = g_k$. Thus if $\partial g_k = 0$ for all k,

$$A = \Lambda^* E \otimes \mathcal{T} E$$

is an A_∞-algebra.

Proof: There is a homotopy operator $H = H_k \in \mathrm{Hom}^{-1}(A^{\otimes k}, A)$ with

$$\partial H + H\partial = k \cdot \mathrm{Id}.$$

For $\partial = v\wedge$, H is essentially the interior product with $v^\sharp$. Thus

$$g_k = \frac{1}{k}(\partial H + H\partial) g_k = \partial\left(\frac{1}{k} H g_k\right).$$

Main algebraic result

Theorem

If $\partial g_k = 0$*, then there exists an explicit solution to the equation* $\partial m_k = g_k$*. Thus if* $\partial g_k = 0$ *for all* k*,*

$$A = \Lambda^* E \otimes \mathcal{T} E$$

is an A_∞*-algebra.*

Proof: There is a homotopy operator $H = H_k \in \mathrm{Hom}^{-1}(A^{\otimes k}, A)$ with

$$\partial H + H\partial = k \cdot \mathrm{Id}.$$

For $\partial = v\wedge$, H is essentially the interior product with $v^\sharp$. Thus

$$g_k = \frac{1}{k}(\partial H + H\partial) g_k = \partial \left(\frac{1}{k} H g_k \right).$$

Moral: An endomorphism of an inner product space and a choice of a nonzero vector should give rise to an A_∞-algebra on the Fock space. These algebras are isomorphic for different vectors, so we should have an A_∞-algebra associated to an endomorphism.

Final comments

From Brownian motion and a G-bundle E over a Lie group, we produce a product on a fiber E_x which conjecturally has an A_∞-structure. These give deformations of the standard Hopf tensor algebra parametrized by G-connections on E. We should also have A_∞-coproduct structures with compatibility. So we should have an "A_∞-Hopf algebra."

Final comments

From Brownian motion and a G-bundle E over a Lie group, we produce a product on a fiber E_x which conjecturally has an A_∞-structure. These give deformations of the standard Hopf tensor algebra parametrized by G-connections on E. We should also have A_∞-coproduct structures with compatibility. So we should have an "A_∞-Hopf algebra."

First order deformations of A_∞-algebras are characterized by elements of the Hochschild cohomology $HH^*(A, A)$. First order deformations of Hopf algebras are characterized by elements in H^2 of a triple complex.

Final comments

From Brownian motion and a G-bundle E over a Lie group, we produce a product on a fiber E_x which conjecturally has an A_∞-structure. These give deformations of the standard Hopf tensor algebra parametrized by G-connections on E. We should also have A_∞-coproduct structures with compatibility. So we should have an "A_∞-Hopf algebra."

First order deformations of A_∞-algebras are characterized by elements of the Hochschild cohomology $HH^*(A, A)$. First order deformations of Hopf algebras are characterized by elements in H^2 of a triple complex.

Questions: (1) What cohomology theory do our deformations lie in?

(2) Which path integrals can be treated by this method? Maybe QM?

Final comments

From Brownian motion and a G-bundle E over a Lie group, we produce a product on a fiber E_x which conjecturally has an A_∞-structure. These give deformations of the standard Hopf tensor algebra parametrized by G-connections on E. We should also have A_∞-coproduct structures with compatibility. So we should have an "A_∞-Hopf algebra."

First order deformations of A_∞-algebras are characterized by elements of the Hochschild cohomology $HH^*(A, A)$. First order deformations of Hopf algebras are characterized by elements in H^2 of a triple complex.

Questions: (1) What cohomology theory do our deformations lie in?

(2) Which path integrals can be treated by this method? Maybe QM? Maybe not

Final comments

From Brownian motion and a G-bundle E over a Lie group, we produce a product on a fiber E_x which conjecturally has an A_∞-structure. These give deformations of the standard Hopf tensor algebra parametrized by G-connections on E. We should also have A_∞-coproduct structures with compatibility. So we should have an "A_∞-Hopf algebra."

First order deformations of A_∞-algebras are characterized by elements of the Hochschild cohomology $HH^*(A, A)$. First order deformations of Hopf algebras are characterized by elements in H^2 of a triple complex.

Questions: (1) What cohomology theory do our deformations lie in?

(2) Which path integrals can be treated by this method? Maybe QM? Maybe not

Thank you!