Non-generic properties of Optimizing measures for Continuous Functions

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Setting

X phase space (topological space / measurable space)

$T : X \to X$ law of time evolution (continuous map / measurable map)

(X, T) (topological / measurable) dynamical system

$\mathcal{O}(x) := \{T^n(x) : n \in \mathbb{S}\}$ orbit

where $T^n = T \circ T \circ \cdots \circ T$ and $\mathbb{S} = \mathbb{N}$ or $\mathbb{Z}$

if the map is invertible

- **Study properties of "typical" orbits**
- **Evaluate complexity of dynamical systems**

Example 1-1

Rotation Maps

$$\alpha \in [0,1)$$

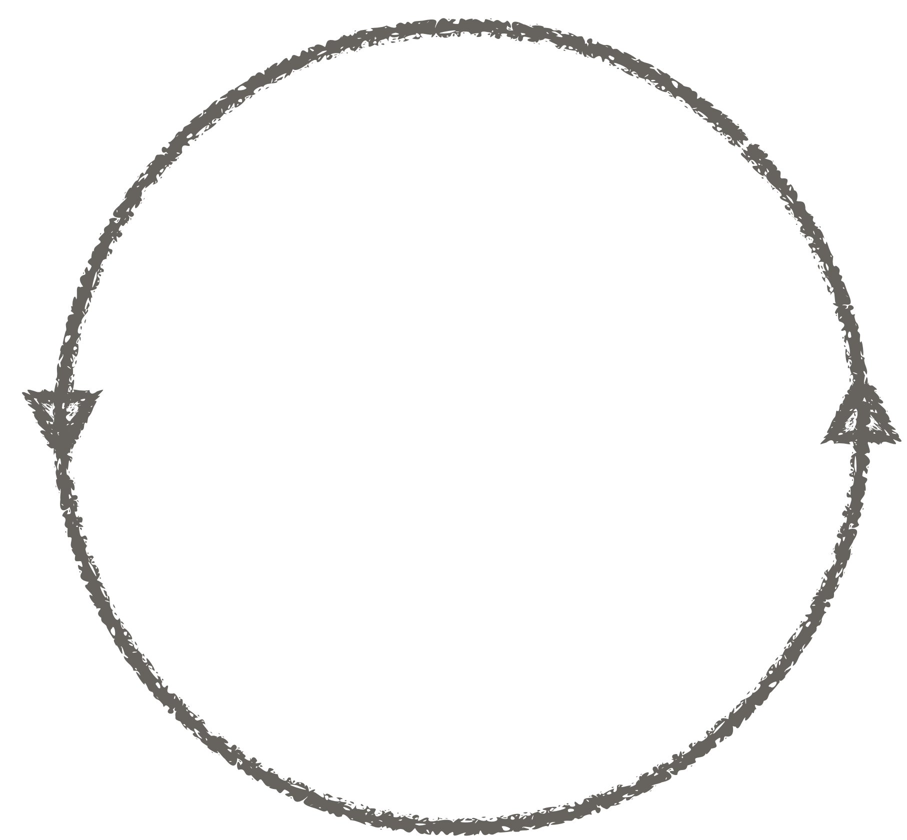
$$T : S^1 \to S^1$$

$$T(x) = x + \alpha \pmod 1$$

α : rational

$^{\exists} n \geq 1$ s.t. $T^n(x) = x$

Every point is periodic.



α : irrational

There are no periodic points.
Every orbit is dense.

The distance between two points doesn't change.

Example 1-2

Expanding Maps

There is a dense orbit.

Periodic points are dense.

The distance of two points expands exponentially as time goes to infinity.

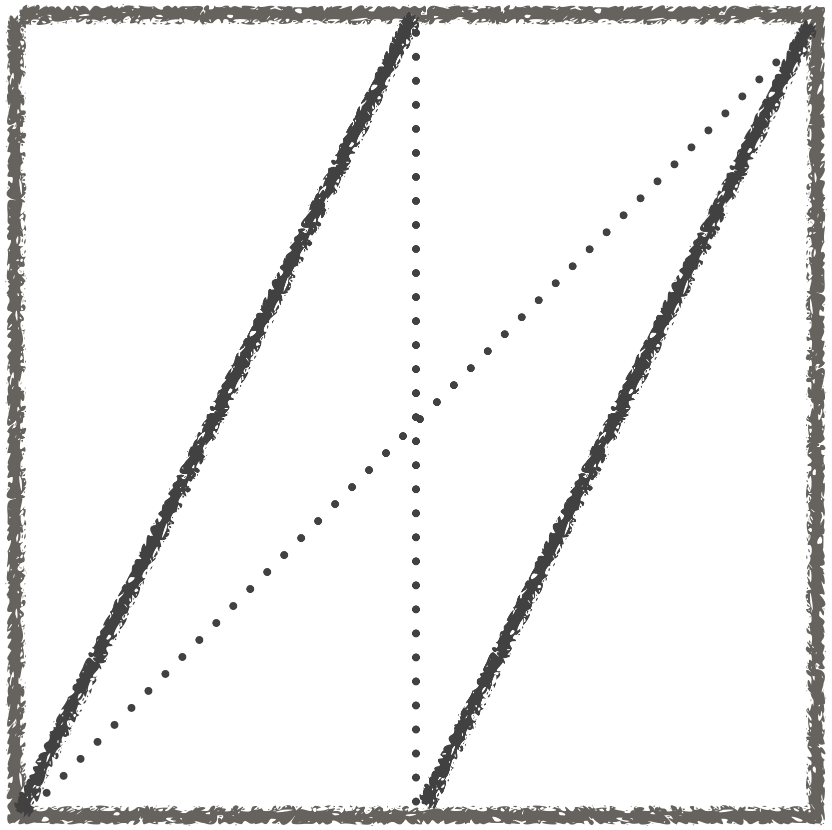
In the case of the left picture, “the rate of the expansion” is constant .

$$T : S^1 \to S^1$$

$$T(x) = 2x \pmod 1$$

More generally,
the C^1 self-map on the circle is expanding if

$$\min_{x \in S^1} |DT(x)| > 1$$



Example 1-3

Piecewise Linear Markov Maps

There is a dense orbit.

Periodic points are dense.

The distance of two points expands exponentially as time goes to infinity.

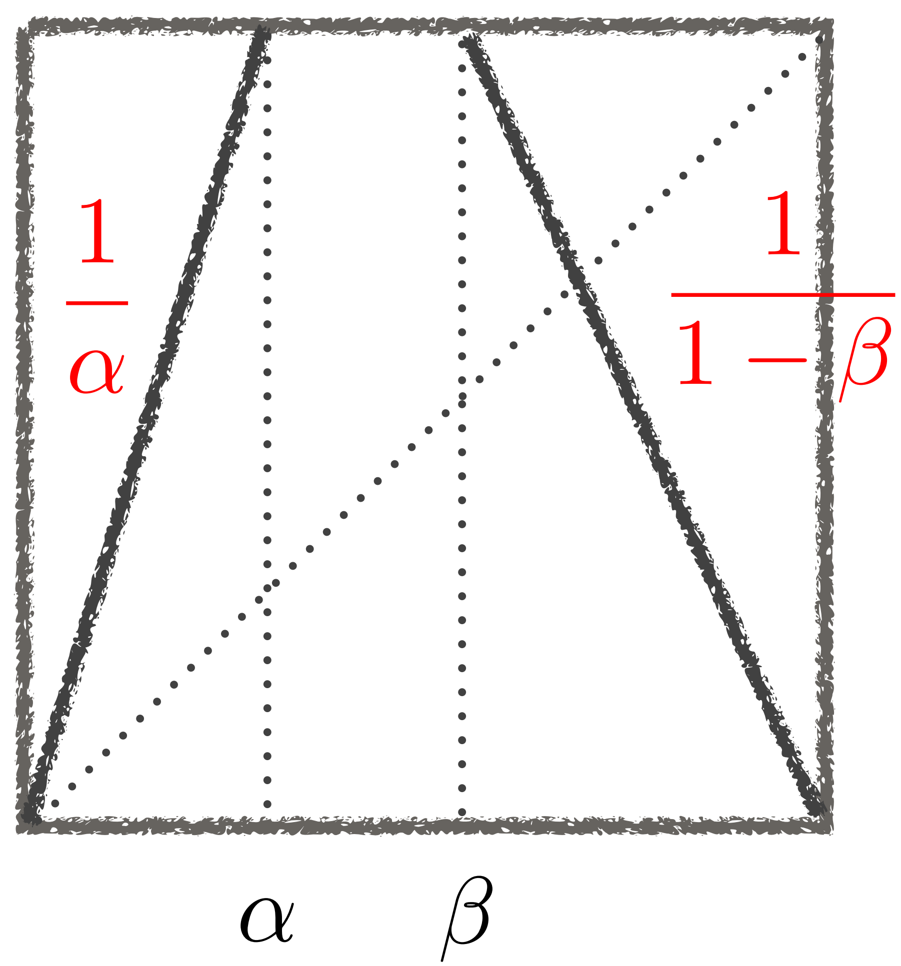
The rate of the expansion dramatically changes, depending on initial points.

$$\Omega := [0,\alpha] \cup [\beta, 1]$$
$$T : \Omega \to [0,1]$$

We restrict our attention into the set of points which do not escape from Ω

$$\Lambda := \bigcap_{k=0}^{\infty} T^{-k}\Omega$$ Cantor set

$$T : \Lambda \to \Lambda$$



the product topology of the discrete topology

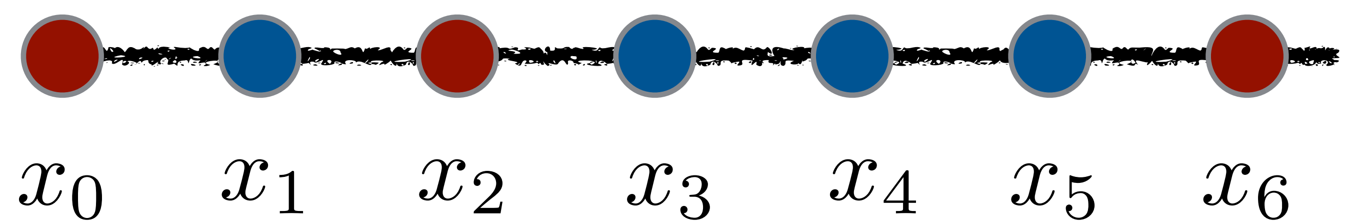
$$\sigma : \{r, b\}^{\mathbb{N}} \to \{r, b\}^{\mathbb{N}}$$ **full shift**

$$\sigma(\{x_i\}_{i=0}^{\infty}) = \{x_{i+1}\}_{i=0}^{\infty}$$

Example 1-4

Subshift of Finite Type

$$\omega \in \{r, b\}^{\mathbb{N}} \quad \xleftarrow{\sigma}$$



There is a dense orbit.

Periodic points are dense.

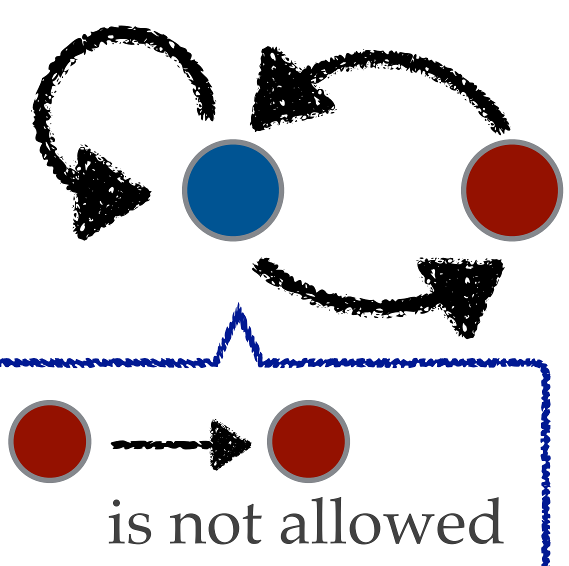
The full shift is topologically conjugate with the previous example, piecewise linear Markov map.

One solution to understand dynamical systems is to find "coding" to subshifts of finite type.

Defining allowable words by the graph, we obtain a subshift of finite type

$$\sum_A \subset \{r, b\}^{\mathbb{N}}$$

subshift of finite type



Ergodic Optimization

X compact metric space

$T : X \to X$ continuous map

$\phi : X \to \mathbb{R}$ continuous function

$$S_n\phi(x) := \sum_{k=0}^{n-1} \phi(T^k(x))$$ where $n \in \mathbb{N}$ dynamical sum

- **We are interested in the time average as time goes to infinity**

$$\overline{\phi}(x) := \lim_{n\to\infty} \frac{1}{n} S_n\phi(x)$$ if it exists

- **Find orbits maximizing the time average**

Example 2-1

Hitting Frequency

$A \subset X$ clopen subset

$$\phi(x) := 1_A(x)$$ characteristic function

$$= \begin{cases} 1 & \text{if} \quad x \in A \\ 0 & \text{o.w.} \end{cases}$$

$$\frac{1}{n} S_n \phi(x)$$

$$= \frac{1}{n} \#\{0 \leq k \leq n-1 : T^k(x) \in A\}$$

Hitting Frequency

The time average of the characteristic function represents the frequency with which an orbit hits A.

Example 2-2

Lyapunov Exponent

The time average of the geometric function represents the exponential expanding rate along with an orbit.

$T : S^1 \to S^1$ expanding map i.e.

$$\min_{x \in S^1} |DT(x)| > 1$$

$\phi(x) := \log |DT(x)|$ geometric function

$$\frac{1}{n} S_n \phi(x) = \frac{1}{n} \sum_{k=0}^{n-1} \log |DT(T^k(x))|$$

$$= \frac{1}{n} \log |DT^n(x)|$$

Lyapunov Exponent

Maximizing Measures

$\mathcal{M}_T$ the space of invariant Borel probability measures with the weak*-topology

$\mu(T^{-1}B) = \mu(B)$ for every Borel subsets

For a continuous function $\phi : X \to \mathbb{R}$

$$\sup_{x \in X} \limsup_{n \to \infty} \frac{1}{n} S_n \phi(x) = \max_{\nu \in \mathcal{M}_T} \int \phi \, d\nu$$

We call invariant measures attaining the maximum **maximizing measures**

For "ergodic" maximizing measure, by Birkhoff's ergodic theorem,

$$\lim_{n \to \infty} \frac{1}{n} S_n \phi(x) = \int \phi \, d\mu \qquad \mu - a.e.$$

We are interested in uniqueness, support and entropy

- **Study properties of maximizing measures**

Generic, Non-generic Properties

We are interested in properties of maximizing measures for "many" functions

$$C(X) := \{\phi : X \to \mathbb{R} : \ \text{continuous}\}$$

The space of continuous functions with the supremum norm

A property $\mathcal{P}$ is **generic** if

$\{\phi \in C(X) : \mathcal{P} \text{ holds for } \phi\}$ contains a residual subset

the intersection of countably many open dense sets

Generic Properties

Theorem (Bousch, Brémont, Jenkinson, Morris)

Assume T has the specification property.

Examples 1-2,3,4 satisfy this condition

There exists a residual subset $\mathcal{R}$ of $C(X)$ such that for every $\phi \in \mathcal{R}$ there exists

- **G1** a unique maximizing measure
- **G2** and its has full support
- **G3** and zero entropy.

Main Results

(Non-Generic Properties)

We also pay attention to the functions outside the residual subset.
Main results show non-generic properties of maximizing measures.

$\mathcal{M}_e$ the set of ergodic measures

Theorem A

Assume $\mathcal{M}_e$ is arcwise-connected.
There exists a dense subset $\mathcal{D}$ of $C(X)$ such that for every $\phi \in \mathcal{D}$ there exists

D1 uncountably many ergodic maximizing measures.

Theorem B

Consider a subshift of finite type $(\sum_A, \sigma)$.
There exists a dense subset $\mathcal{D}$ of $C(\sum_A)$ such that for every $\phi \in \mathcal{D}$

- **D1** uncountably many ergodic maximizing measures
- **D2** with full support
- **D3** and positive entropy.

References

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