Non-generic properties of optimizing measures for continuous functions

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What is Ergodic Optimization?

(X, T) dynamical system
X compact metric space (phase space)
$T : X \to X$ continuous map (law of time evolution)

For a continuous "performance" function $\phi : X \to \mathbb{R}$ we consider its time average. Define

$$\overline{\phi}(x) := \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \phi \circ T^k(x),$$

if it exists.

Problem 1

Fined orbits maximizing the time average $\overline{\phi}$.

Why is the time average interesting?

- Full shift

$X = \{0,1\}^{\mathbb{N}}$
$\sigma : X \to X$ defined by $\sigma(\{x_i\}) = \{x_{i+1}\}$

Example 2 (Hitting Frequency)

$A \subset X$ clopen subset e.g. $A := [0] = \{\{x_i\} \in X : x_0 = 0\}$
$\phi = \chi_A$ the characteristic function of A

$$\begin{aligned}\overline{\phi}(x) &= \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \chi_A(T^k(x)) \\ &= \lim_{n\to\infty} \frac{1}{n} \# \left\{ k \in \{0,1,\ldots,n-1\} : T^k(x) \in A \right\}\end{aligned}$$

- The time average $\overline{\phi}$ represents the hitting frequency for the set A.

Why is the time average interesting?

- Expanding Map

$S^1 = \mathbb{R}/\mathbb{Z}$ the circle
$T : S^1 \to S^1$ with $\min_{x \in S^1} |DT(x)| > 1$ e.g $T(x) = 2x \bmod 1$

Example 3 (Lyapunov Exponent)

$\phi(x) = \log |DT(x)|$ geometric function

$$\overline{\phi}(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \log |DT(T^k x)| = \lim_{n\to\infty} \frac{1}{n} \log |DT^n(x)|$$

- The time average $\overline{\phi}$ coincides with the Lyapunov exponent.

How do we find maximizing orbits?

$\mathcal{M}_T$ the space of all invariant Borel probability measures

Proposition 4

For all continuous function ϕ on X we have

$$\sup_{x \in X} \limsup_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \phi \circ T^k(x) = \max_{\nu \in \mathcal{M}_T} \int \phi \, d\nu$$

- $\mu \in \mathcal{M}_T$ attaining the maximum is called a maximizing measure.

Problem 5

Study the properties of maximizing measures.

What does "generic" mean?

We are interested in the properties of maximizing measures for "many" functions.

$C(X)$ the space of continuous functions with the supremum norm

Definition 6

A property $\mathcal{P}$ is generic if
$\{\phi \in C(X) : \mathcal{P} \text{ holds for } \phi\}$ contains a residual set.

Uniqueness

- Generic property

Theorem 7 (Jenkinson)

There exists a residual subset $\mathcal{R}$ of $C(X)$ such that for every $\phi \in \mathcal{R}$ there exists

G1 *a unique maximizing measure*

- Non-generic property

$\mathcal{M}_e$ the set of all ergodic Borel probability measures on X

Theorem A

Assume $\mathcal{M}_e$ is arcwise-connected.
Then there exists a dense subset $\mathcal{D}$ of $C(X)$ such that for every $\phi \in \mathcal{D}$ there exist

D1 *uncountably many ergodic maximizing measures.*

Support and Entropy

- Generic

Theorem 8 (Bousch, Brémont, Jenkinson, Morris)

Assume T satisfies the specification property.
There exists a residual subset $\mathcal{R}$ of $C(X)$ such that for every $\phi \in \mathcal{R}$ there exists

G1 *a unique maximizing measure*

G2 *and it is fully supported*

G3 *and has zero entropy.*

Main Theorems

- Non-generic

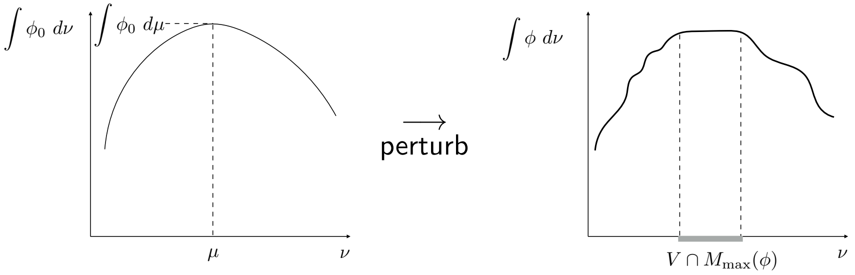
Theorem B

Let (X, T) be a topologically mixing subshift of finite type. Then there exists a dense subset $\mathcal{D}$ of $C(X)$ such that for every $\phi \in \mathcal{D}$ there exist

- D1 *uncountably many ergodic maximizing measures*
- D2 *which are full support*
- D3 *and have positive entropy.*

Idea of the proof

Take $\phi_0 \in C(X)$ and consider the graph of $\nu \mapsto \int \phi_0 d\nu$.



The perturbation is ensured by the Bishop Phelps theorem.

Thank you for your attention

Tangency to the Convex Functional Q

Define $Q : C(X) \to \mathbb{R}$ by

$$Q(\phi) = \max_{\nu \in M(X,T)} \int \phi \, d\nu.$$

The functional Q is continuous and convex functional on $C(X)$.
We can characterize maximizing measures by the tangency to Q.

Proposition 9

Let $\phi \in C(X)$. The following conditions are equivalent.

M1 : $\mu \in M(X,T)$ *is a maximizing measure for* ϕ;

M2 : μ *is tangent to* Q *at* ϕ

Tangency to the Convex Functional Q

$C(X)$ the Banach space with the supremum norm
$Q : C(X) \to \mathbb{R}$ the convex and continuous functional
$\mu : C(X) \to \mathbb{R}$ a signed measure (bounded linear functional)

Definition 10

μ is tangent to Q at $\phi \in C(X)$ if

$$\mu(\psi) \leq Q(\phi + \psi) - Q(\phi)$$

for all $\psi \in C(X)$.

Bishop Phelps Theorem

Theorem (Bishop Phelps)

For every $\phi_0 \in C(X)$ and $\varepsilon > 0$ there exits $\phi \in C(X)$ and a signed measure μ which is a ϕ-maximizing measure such that

$$\|\mu - \mu_0\| \leq \varepsilon \quad \|\phi - \phi_0\| \leq \frac{1}{\varepsilon}\left(Q(\phi_0) - \mu_0(\phi_0)\right)$$

Support and Entropy

Theorem 11 (Contreras)

Let T be "expanding".
Then there exists a open and dense subset $\mathcal{O}$ of $Lip(X)$ such that for every $\phi \in \mathcal{O}$ there exists

- O1 *a unique maximizing measure*
- O2 *and its has periodic support.*

Idea of the Proof

It is known that for every $\mu \in \mathcal{M}(X, T)$
there exits a unique Borel probability measure α on $\mathcal{M}_e(X, T)$ such that

$$\mu(\phi) = \int_{\mathcal{M}_e(X,T)} m(\phi) \; d\alpha(m)$$

for all $\phi \in C(X)$. We call the α the ergodic decomposition of μ.

Bishop Phelps Theorem

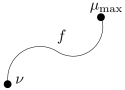
Theorem (Bishop Phelps)

For every $\phi_0 \in C(X)$, $\varepsilon > 0$ and $\mu_0 \in \mathcal{M}(X,T)$ there exits $\phi \in C(X)$ and a ϕ-maximizing measure μ such that

$$\|\mu - \mu_0\| \leq \varepsilon \quad \|\phi - \phi_0\| \leq \frac{1}{\varepsilon}\left(\max_{\nu \in \mathcal{M}(X,T)} \int \phi_0 \, d\nu - \mu_0(\phi_0)\right)$$

Sketch of the Proof

Let $\phi_0 \in C(X)$ and $\varepsilon > 0$. Pick $\mu_{\max} \in \mathcal{M}(\phi_0)$.
There exists $f : [0,1] \to \mathcal{M}_e(X,T)$ such that $f(0) = \mu_{\max}$ and $f(1) = \nu$. Define $\tilde{\alpha} = f_* \mathrm{Leb}_{[0,1]} \rightsquigarrow$ non-atomic



Restricting $\tilde{\alpha}$ to the set
$\left\{\nu \in \mathcal{M}_e(X,T) : \max_{\nu \in \mathcal{M}(X,T)} \int \phi_0 \, d\nu - \varepsilon^2 < \int \phi_0 \, d\nu\right\}$,
we have α_0 and $\mu_0 = \int_{\mathcal{M}_e(X,T)} m \, d\alpha_0(m)$.

By the Bishop Phelps Theorem,
we have ε-close $\phi \in C(X)$ and $\mu \in \mathcal{M}(\phi)$.

Let α be the mass distribution of μ.

$$\alpha_0(\mathrm{supp}\ \alpha) > 0 \quad \text{and} \quad \mathrm{supp}\ \alpha \subset \mathcal{M}(\phi)$$

Then $\mathrm{supp}\ \alpha$ contains uncountably many points, which prove the theorem.