K-Theory for Locally Compact Spaces

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Definition of Vect(X)

Let X be a topological space.

- The collection of isomorphism classes $\langle\xi\rangle$ of complex vector bundles ξ over X is denoted Vect(X).
- Addition on Vect(X) is defined by $\langle\xi\rangle + \langle\eta\rangle = \langle\xi \oplus \eta\rangle$, where $\xi \oplus \eta := \{(v_1, v_2) \in V_1 \times V_2 \mid \pi_1(v_1) = \pi_2(v_2)\}$ is the direct sum of two vector bundles $\xi = (V_1, \pi_1)$ and $\eta = (V_2, \pi_2)$.

Definition of $K^{-n}(X)$

- Let X be a compact Hausdorff space. Define $K^0(X)$ be the Grothendieck group of $(\mathrm{Vect}(X), +)$.
- Let X be a locally compact Hausdorff Space and let X^+ be the one-point compactification of X. Define $K^0(X)$ be $\mathrm{Ker}(j^* : K^0(X^+) \to K^0(pt) \cong \mathbb{Z})$, where the group homomorphism j^* is induced by the inclusion map $j : pt \to X^+$.
- Define $K^{-n}(X) = K^0(X \times \mathbb{R}^n)$ for every natural number n and locally compact Hausdorff space X.

Remark

- The definition of K^0-groups for locally compact spaces is an extension of that for compact spaces.

The Long Exact Sequence

Let A be a closed subspace of a locally compact Hausdorff space X, and let $j : A \to X$ be the inclusion map. Then, for each natural number n there exists a natural homomorphism $\delta : K^{-n-1}(X) \to K^{-n}(X \backslash A)$ that makes the infinite sequence

$$\cdots \to K^{-2}(X) \xrightarrow{j^*} K^{-2}(A) \xrightarrow{\delta}$$
$$K^{-1}(X\backslash A) \to K^{-1}(X) \xrightarrow{j^*} K^{-1}(A) \xrightarrow{\delta} K^0(X\backslash A) \to K^0(X) \xrightarrow{j^*} K^0(A)$$

exact.

Bott Periodicity

For every locally compact Hausdorff space X, there exists a natural isomorphism

$$\beta : K^0(X) \to K^{-2}(X).$$

K-Groups of $\mathbb{R}^n$ and S^n

Fact

- $K^0(pt) \cong \mathbb{Z}$ and $K^{-1}(pt) \cong 0$.

Bott periodicity gives us

$$K^0(\mathbb{R}^n) \cong K^0((pt) \times \mathbb{R}^n) = K^{-n}(pt) \cong \begin{cases} \mathbb{Z} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

This yields

$$K^{-1}(\mathbb{R}^n) = K^0(\mathbb{R}^{n+1}) \cong \begin{cases} 0 & \text{if } n \text{ is even} \\ \mathbb{Z} & \text{if } n \text{ is odd.} \end{cases}$$

Fact

- Let X^+ be the one-point compactification of a locally compact space X, then $K^0(X^+) \cong K^0(X) \oplus \mathbb{Z}$ and $K^{-1}(X^+) \cong K^{-1}(X)$.

Since S^n is the one-point compactification of $\mathbb{R}^n$, we obtain

$$K^0(S^n) \cong K^0((\mathbb{R}^n)^+) \cong K^0(\mathbb{R}^n) \oplus \mathbb{Z} \cong \begin{cases} \mathbb{Z}^2 & \text{if } n \text{ is even} \\ \mathbb{Z} & \text{if } n \text{ is odd.} \end{cases}$$

Similarly,

$$K^{-1}(S^n) \cong K^{-1}((\mathbb{R}^n)^+) \cong K^{-1}(\mathbb{R}^n) \cong \begin{cases} 0 & \text{if } n \text{ is even} \\ \mathbb{Z} & \text{if } n \text{ is odd.} \end{cases}$$

The Six-Term Exact Sequence

Let A be a closed subspace of a locally compact space X, and let $j : A \to X$ be the inclusion map. Then there exists an exact sequence

$$\begin{array}{ccccc} K^0(X\backslash A) & \text{—} & K^0(X) & \xrightarrow{j^*} & K^0(A) \\ \delta \big| & & & & \big| \delta \\ K^{-1}(A) & \xrightarrow{j^*} & K^{-1}(X) & \text{—} & K^{-1}(X\backslash A) \end{array}$$

with natural connecting maps δ.

Definition of $\mathbb{R}P^n$

$\mathbb{R}P^n$ is defined as the quotient space $S^n/\sim$, where $x \sim y \iff x = \pm y$, equipped with the quotient topology.

Remark

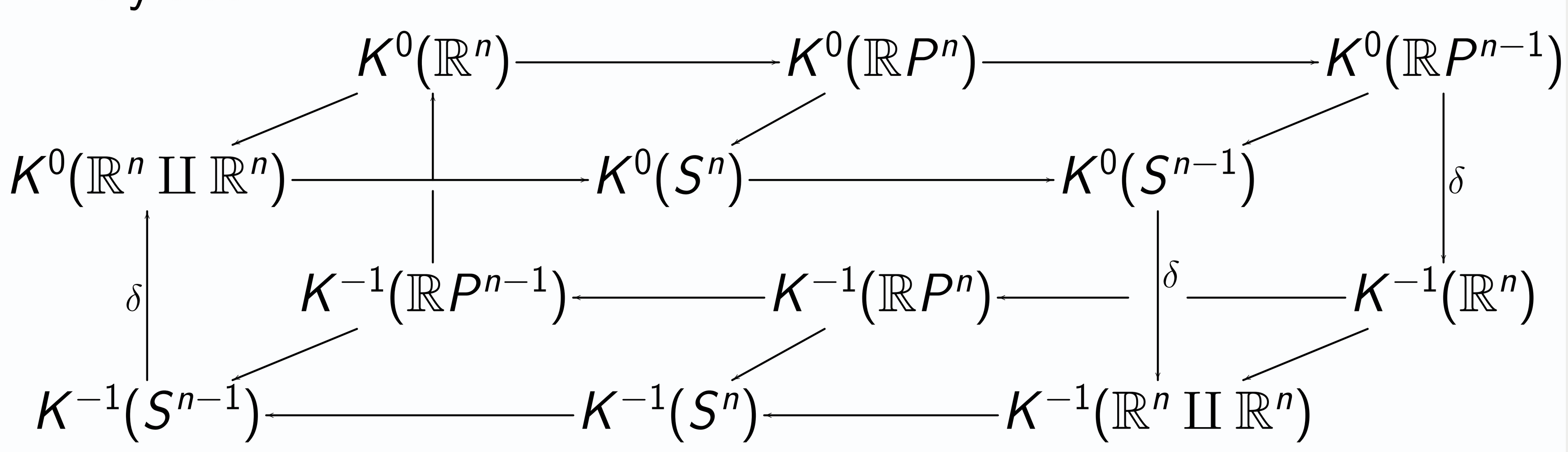
- For each natural number n, $\mathbb{R}P^{n-1}$ is a closed subset of $\mathbb{R}P^n$ and $\mathbb{R}P^n \backslash \mathbb{R}P^{n-1} \cong \mathbb{R}^n$

K-Groups of $\mathbb{R}P^n$

Consider the following commutative diagram

$$\begin{array}{ccc} \mathbb{R}P^n & \text{—} & \mathbb{R}P^{n-1} \\ | & & | \\ S^n & \text{—} & S^{n-1}. \end{array}$$

This yields



The naturality of the connecting maps δ makes the above diagram commutative.
By the diagram chase, inductively we obtain

$$K^0(\mathbb{R}P^n) \cong \begin{cases} \mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^{\frac{n}{2}} & \text{(if } n \text{ is even)} \\ \mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^{\frac{n-1}{2}} & \text{(if } n \text{ is odd)} \end{cases}$$

and

$$K^{-1}(\mathbb{R}P^n) \cong \begin{cases} 0 & \text{(if } n \text{ is even)} \\ \mathbb{Z} & \text{(if } n \text{ is odd).} \end{cases}$$

Definition of $\mathbb{C}P^n$

$\mathbb{C}P^n$ is defined as the quotient space $(\mathbb{C}^{n+1} \backslash \{0\})/\sim$, where $x \sim y \iff x = \lambda y$ for some complex number λ, equipped with the quotient topology.

Remark

- For each natural number n, $\mathbb{C}P^{n-1}$ is a closed subset of $\mathbb{C}P^n$ and $\mathbb{C}P^n \backslash \mathbb{C}P^{n-1} \cong \mathbb{C}^n$

K-Groups of $\mathbb{C}P^n$

Considering the inclusion

$$\mathbb{C}P^n \hookleftarrow \mathbb{C}P^{n-1},$$

we have the exact sequence

$$\begin{array}{ccccc} K^0(\mathbb{C}^n) & \text{—} & K^0(\mathbb{C}P^n) & \text{—} & K^0(\mathbb{C}P^{n-1}) \\ | & & & & | \\ K^{-1}(\mathbb{C}P^{n-1}) & \text{—} & K^{-1}(\mathbb{C}P^n) & \text{—} & K^{-1}(\mathbb{C}^n). \end{array}$$

This yields

$$K^0(\mathbb{C}P^n) \cong \mathbb{Z}^{n+1}$$

and

$$K^{-1}(\mathbb{C}P^n) \cong 0.$$

References

- Park, Efton. *Complex topological K-theory*. Vol. 111. Cambridge University Press, 2008.
- Rørdam, Mikael, Flemming Larsen, and Niels Laustsen. *An Introduction to K-theory for C*-algebras*. Vol. 49. Cambridge University Press, 2000.