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A note on singular points of bundle homomorphisms between a tangent distribution into a vector bundle of the same rank

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Introduction

In [5, 7], the notion of coherent tangent bundle is introduced. It is a bundle homomorphism between the tangent bundle and a vector bundle with the same rank with a kind of metric. This is a generalization of fronts and C^∞-maps between the same dimensional manifolds. Singular points of bundle homomorphisms $\varphi : TM \to E$ are points where $\varphi(p) : T_pM \to E_p$ is not a bijection. In [5, 7], differential geometric invariants of singularities of bundle homomorphisms are defined and investigated. On the other hand, in [8], topological properties of singular sets of bundle homomorphisms without metric are studied. See [1] for another kind of application of coherent tangent bundle. In this poster, we consider rank $r(< m)$ tangent distributions instead of the tangent bundles of m-dimensional manifolds. Since $r < m$, the singularities appearing on the bundle homomorphisms are slightly different from the case $\varphi : TM \to E$, where $\dim M = \operatorname{rank} E = m$, and the case $\varphi : TM \to E$, where $\dim M = \operatorname{rank} E = r$ either.

Let D_1 be a rank r tangent distribution on an m-dimensional manifold M. Let N be an r dimensional manifold, and $f : M \to N$ a map. Then a bundle homomorphism $\varphi = df : D_1 \to f^*TN$ is induced from f. Singularities of φ should be related to D_1 and f. Here we stick to our interest into the low dimensional case, we study the relationships when f is a Morin map, and D_1 is the foliation or the contact structure when $m = 3$, $r = 2$.

1 Bundle homomorphisms and their singular point

With the terminology of [7], we give definition of singular points of bundle homomorphisms. We set

- M : m-manifold, D_1 : rank r ($r < m$) tangent distribution of M, D_2 : a rank r vector bundle over M,
- φ: $D_1 \to D_2$ be a bundle homomorphism.

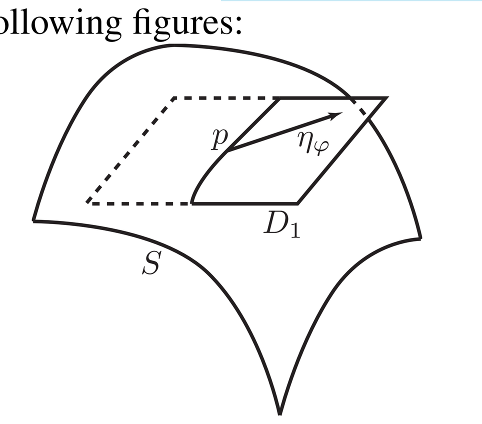
A point $p \in M$ is called *singular point* of φ if $\operatorname{rank} \varphi_p < r$.

Lemma 1.1 ([9]). If p is a corank one singular point of φ. Then there exists a neighborhood U of p and a section $\eta_\varphi \in \Gamma(D_1)$ such that if $q \in S \cap U$ then $(\eta_\varphi)_q$ is a generator of the kernel of φ_q.

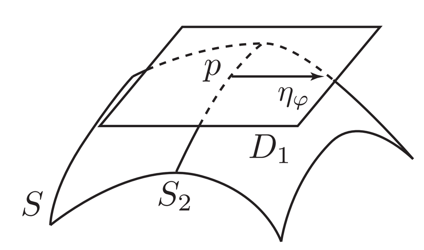
We call η_φ the *null section* of φ. We set $\lambda_\varphi = \det M_\varphi$. We call $p \in S$ is *non-degenerate* if $d\lambda_\varphi(p) \neq 0$. The notions of the null section and the non-degeneracy is introduced in [2]. It is shown that non-degenerate singular points are of corank one. Since $S = \{\lambda_\varphi(p) = 0\}$, S is a codimension one submanifold near a non-degenerate singular point. With the terminology of [3, 6], we give the following definition:

Definition 1.2 ([9]). We call a singular point $p \in S$ is a *fold-like singular point* if it is corank one, and $\eta_\varphi\lambda_\varphi(p) \neq 0$. We call $p \in S$ is a *cusp-like singular point* if p is non-degenerate and $\eta_\varphi\lambda_\varphi(p) = 0$ and $\eta_\varphi^2\lambda_\varphi(p) \neq 0$. We call $p \in S$ is a *swallowtail-like singular point* if p is non-degenerate, and $\eta_\varphi\lambda_\varphi(p) = \eta_\varphi^2\lambda_\varphi(p) = 0$ and $\operatorname{rank} d(\lambda_\varphi, \eta_\varphi\lambda_\varphi, \eta_\varphi^2\lambda_\varphi) = 3$ at p.

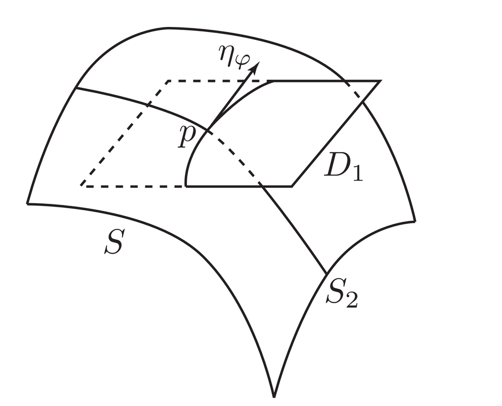
If p is a fold-like singular point, and $(D_1)_p = T_pS$, then $(\eta_\varphi)_p \in T_pS$. Thus $(D_1)_p \neq T_pS$. Let p be a cusp-like singular point. If $e_1\lambda_\varphi = e_2\lambda_\varphi = 0$ at p, then $(D_1)_p = T_pS$. In this case, we call p *cusp-like singular point of tangent type*. If $(e_1\lambda_\varphi, e_2\lambda_\varphi) \neq (0, 0)$ at p, then $(D_1)_p$ is transversal to T_pS. In this case, we call p *cusp-like singular point of transverse type*. The picture of S and D_1 can be drawn in the following figures:



fold-like singular point



cusp-like singular point of tangent type

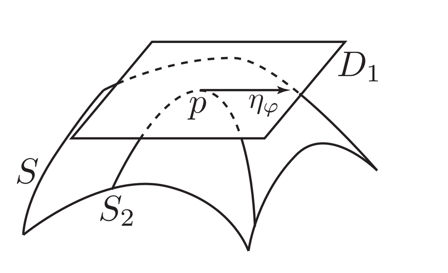


cusp-like singular point of transverse type

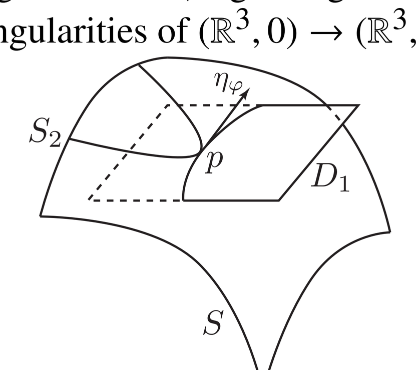
If $p \in S$ is a swallowtail-like singular point, then S_2 is one-dimensional submanifold of S. Let (u, v) be a coordinate system near p of S. Let $\gamma(t) = (\gamma_1(t), \gamma_2(t))$ $(\gamma(0) = p)$ be a parameterization of S_2 with respect to (u, v), and let $\eta_{\gamma(t)} = a(t)\partial_u + b(t)\partial_v$. Then we have the following

Proposition 1.3 ([9]). Let $p \in S$ is a swallowtail-like singular point. We set $\mu(t) = \begin{pmatrix} \gamma_1(t) & a(t) \\ \gamma_2(t) & b(t) \end{pmatrix}$. Under the above notation, it holds that $\mu(0) = 0, \mu'(0) \neq 0$.

Swallowtail-like singular point also has tangent and transverse types. If $e_1\lambda_\varphi = e_2\lambda_\varphi = 0$ at p, then $(D_1)_p = T_pS$. We call p a *swallowtail-like singular point of tangent type* in this case. If $(e_1\lambda_\varphi, e_2\lambda_\varphi) \neq (0, 0)$ at p, then $(D_1)_p$ is transversal to T_pS. In this case, we call p a *swallowtail-like singular point of transverse type* (see the figures below.) Ignoring arrangements of D_1, relationship of S, S_2 and η_φ is similar to that of the Morin singularities of $(\mathbb{R}^3, 0) \to (\mathbb{R}^3, 0)$ ([6]).



swallowtail-like singular point of tangent type



swallowtail-like singular point of transverse type

2 Generic singularities

We see the generic singularities of φ is fold-like, cusp-like and swallowtail-like singular points if $m = 3$ and $r = 2$. The bundle homomorphism φ can be regarded as a section of the homomorphism bundle $\operatorname{Hom}(D_1, D_2)$. We set $E = \operatorname{Hom}(D_1, D_2)$. Since the set of sections $\Gamma(E)$ is a subset of $C^\infty(M, E)$, we derive the Whitney C^∞ topology to $\Gamma(E)$.

Proposition 2.1 ([9]). Assume that $m = 3$ and $r = 2$. Then the set

$$\{\varphi \in \Gamma(E) \mid \text{any } p \in S \text{ is fold-like, cusp-like or swallowtail-like}\}$$

is dense.

3 Morin singularities from a manifold with a distribution

Let N be an r-dimensional manifold and $f \colon M \to N$ a map. Setting $D_2 = f^*TN$ we obtain a bundle homomorphism $\varphi : D_1 \to D_2$, which is called a *bundle homomorphism induced by f*, by $\varphi(v) = df(v)$. In this section, assuming f be a Morin singularity, we see relationships of φ, D_1 and f in the case of $m = 3$, $r = 2$. Since we consider local cases, we regard a map $f \colon M \to N$ as a map germ $f : (\mathbb{R}^3, 0) \to (\mathbb{R}^2, 0)$.

Morin singularities

We give a belief review on the Morin singularities of $(\mathbb{R}^3, 0) \to (\mathbb{R}^2, 0)$. The map-germ $f, g \colon (\mathbb{R}^3, 0) \to (\mathbb{R}^2, 0)$ are said to be $\mathcal{A}$-equivalent (which is denoted by $f \overset{\mathcal{A}}{\sim} g$) if there exist diffeomorphism-germs $\sigma : (\mathbb{R}^m, 0) \to (\mathbb{R}^m, 0)$ and $\tau : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ such that $\tau \circ f \circ \sigma^{-1} = g$.

Definition 3.1 ([4]). • The map-germ $f : (\mathbb{R}^3, 0) \to (\mathbb{R}^2, 0)$ is called a *definite fold* (respectively, a *indefinite fold*) if $f \overset{\mathcal{A}}{\sim} (u, v, w) \mapsto (u, v^2 + w^2)$ (respectively, $(u, v^2 - w^2)$) at 0.
• The map-germ $f : (\mathbb{R}^3, 0) \to (\mathbb{R}^2, 0)$ is called a *cusp* if the map-germ $f \overset{\mathcal{A}}{\sim} (u, v, w) \mapsto (u, v^2 + w^3 + uw)$.

Definite fold, indefinite fold and cusp are called Morin singularities, and it is known that generic singularities appearing on maps from a 3-manifold to a 2-manifold are only Morin singularities. A characterization of Morin singularities is given as follows: Let $f : (\mathbb{R}^3, 0) \to (\mathbb{R}^2, 0)$ be a map-germ and $\operatorname{rank} df_0 = 1$. Then there exists a tuple of vector fields $\{\xi, \eta_1, \eta_2\}$ such that

$$\langle \xi(0), \eta_1(0), \eta_2(0) \rangle = T_0\mathbb{R}^3, \quad \langle \eta_1, \eta_2 \rangle = \ker df_p, \quad p \in S(f),$$

where $S(f)$ is the set of singular points of f. We set

$$\lambda_1 = \det(\xi f, \eta_1 f), \quad \lambda_2 = \det(\xi f, \eta_2 f), \quad H = \begin{pmatrix} \eta_1\lambda_1 & \eta_2\lambda_1 \\ \eta_1\lambda_2 & \eta_2\lambda_2 \end{pmatrix}.$$

Then f at 0 is a definite fold (respectively, indefinite fold) if and only if $\det H(0) > 0$ (respectively, $\det H(0) < 0$). We assume that $\operatorname{rank} H(0) = 1$, then there exists a vector field $\theta = a_1\eta_1 + a_2\eta_2$ on $S(f)$ such that $\langle \theta_0 \rangle = \ker H(0)$. Then f at 0 is a cusp if and only if $\theta H(0) \neq 0$. See [4] in detail.

Conditions for singularities

We consider the conditions of singular points of fold-like, cusp-like and swallowtail-like singular points under the assumption that f is regular, fold and cusp since these are generic singularities.

When f is regular at 0, and $D_1 \not\subset \ker df_0$, then φ is non-singular. When f is singular at 0, and $D_1 \subset \ker df_0$, then φ is of rank zero at 0. Since we are stick to rank one singular points of φ, we assume that $D_1 \cap \ker df_0$ is one-dimensional. By taking a suitable local frame $\{e_1, e_2\}$ of D_1, we may assume that $e_1 f(0) \neq 0$. The bundle homomorphism φ can be represented by the matrix

$$(e_1 f, e_2 f)$$

by $\{e_1, e_2\}$ and the trivial frame on $\mathbb{R}^2$. Since $\operatorname{rank} \varphi = 1$ at 0, we take a null section η_φ, and set

$$\lambda_\varphi = \det(e_1 f, e_2 f) = \det(e_1 f, \eta_\varphi f).$$

The following proposition holds.

Proposition 3.2 ([9]). The singular point p of φ is fold-like singular point if and only if $\det(e_1 f, \eta_\varphi^2 f) \neq 0$ at p. A non-degenerate singular point p is cusp-like singular point (respectively, swallowtail-like singular point) if and only if $\det(e_1 f, \eta_\varphi^2 f) = 0$, and $\det(e_1 f, \eta_\varphi^3 f) \neq 0$ at p (respectively, $\det(e_1 f, \eta_\varphi^2 f) = \det(e_1 f, \eta_\varphi^3 f) = 0$, $\det(e_1 f, \eta_\varphi^4 f) \neq 0$, and $d \det(\det(e_1 f, \eta_\varphi f), \det(e_1 f, \eta_\varphi^2 f), \det(e_1 f, \eta_\varphi^3 f)) \neq 0$ at p).

4 Restriction of singularities of φ by singular types of f

We assume that f at 0 is a definite fold singular point. Then $\operatorname{rank}(e_1 f, e_2 f, e_3 f) = 1$ on $S(f)$, where $\{e_1, e_2, e_3\}$ is a frame of $T\mathbb{R}^3$. Thus there exist functions k_1, k_2 such that $e_2 f = k_1 e_1 f, e_3 f = k_2 e_1 f$ on $S(f)$. Taking extensions of k_1, k_2 on $\mathbb{R}^3$, we set

$$\eta_2 = -k_1 e_1 + e_2,\ \eta_3 = -k_2 e_1 + e_3,\ \lambda_2 = \det(e_1 f, e_2 f) = \det(e_1 f, \eta_2 f),\ \lambda_3 = \det(e_1 f, e_3 f) = \det(e_1 f, \eta_3 f).$$

Then we see that η_2 is a null section of φ, and λ_2 is the same as λ_φ. Since f is definite fold,

$$H = \det\begin{pmatrix} \eta_2\lambda_2 & \eta_3\lambda_2 \\ \eta_2\lambda_3 & \eta_3\lambda_3 \end{pmatrix} > 0.$$

In particular, $\eta_2\lambda_2 \neq 0$. Thus φ is fold-like at 0 if $\operatorname{rank} \varphi(0) = 1$.

Next we assume that f at 0 is a cusp singular point. Then we take k_1, k_2, η_2, η_3 and λ_2, λ_3 as above. We assume that φ is not fold-like, namely, $\eta_2\lambda_2(0) = 0$. Then since f is cusp,

$$H(0) = \det\begin{pmatrix} \eta_2\lambda_2 & \eta_3\lambda_2 \\ \eta_2\lambda_3 & \eta_3\lambda_3 \end{pmatrix}(0) = 0.$$

Since $\eta_3\lambda_2(0) = \eta_2\lambda_3(0)$, it holds that $\eta_3\lambda_2(0) = 0$. Hence the kernel of H is $\theta = \eta_1$ at 0. Then f is cusp if and only if

$$\eta_1^2\lambda_1(0)\, \eta_2\lambda_2(0) \neq 0.$$

Thus φ is non-degenerate and not fold-like at 0, then φ is cusp-like at 0.

The case D_1 is a foliation

Here we assume D_1 is a foliation. By taking a coordinate system (x, y, z) on $\mathbb{R}^3$, we may assume $D_1 = \langle e_1, e_2 \rangle = \langle \partial_x, \partial_y \rangle$. Let $L(x, y)$ be the leaf which contains the origin, namely, $L(x, y) = f(x, y, 0)$. We have the following proposition

Proposition 4.1 ([9]). Under the above setting, the following holds: (1) φ is fold-like if and only if L is fold. (2) if φ is non-degenerate, then φ is cusp-like if and only if L is cusp. (3) if φ satisfies that $\operatorname{rank} d(\lambda_\varphi, \eta_\varphi\lambda_\varphi)(0) = 2$, then φ is swallowtail-like if and only if L is swallowtail.

A map-germ $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ is called a *fold* if f is $f \overset{\mathcal{A}}{\sim} (u, v) \mapsto (u, v^2)$ at 0. A map-germ $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ is called a *cusp* (respectively, *swallowtail*) if $f \overset{\mathcal{A}}{\sim} (u, v) \mapsto (u, v^3 + uv)$ at 0 (respectively, $(u, v) \mapsto (u, v^4 + uv)$ at 0). Criteria for these singularities are obtained as follows: Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be a map-germ. We set $\lambda = \det J$, where J is the Jacobian matrix of f. A singular point $p \in S(f)$ is *non-degenerate* if $d\lambda(p) \neq 0$. Then the following holds.

Fact 4.2 ([11, 6, 3]). A singular point p is fold if $\eta\lambda(p) \neq 0$. Moreover, a non-degenerate singular point p is cusp (respectively, swallowtail) if $\eta\lambda(p) = 0$ and $\eta^2\lambda(p) \neq 0$ (respectively, $\eta\lambda(p) = \eta^2\lambda(p) = 0$ and $\eta^3\lambda(p) \neq 0$).

The case D_1 is a contact structure

Here we assume D_1 is a contact structure. Since the Hamilton vector field X associated to λ_φ is contained in D_1 on S, we consider the relationship with the behavior of X and the singularities of φ. We may assume $D_1 = \langle e_1, e_2 \rangle = \langle \partial_x, \partial_y - x\partial_z \rangle$ without loss of generality. Since φ can be expressed by $(f_x, f_y - xf_z)$,

$$\lambda_\varphi = \det(f_x, f_y - xf_z).$$

The Hamilton vector field X associated to λ_φ is

$$X = (\lambda_y - x\lambda_z)\partial_x - \lambda_x\partial_y - (\lambda - x\lambda_x)\partial_z = (\lambda_y - x\lambda_z)e_1 - \lambda_x e_2 - \lambda\partial_z.$$

Since $S = \{\lambda_\varphi = 0\}$ holds, $X_p \in D_1$ is equivalent to $p \in S$. We have the following theorem.

Theorem 4.3 ([9]). If φ has a corank one singular point at p, under the above setting, $p \in S$ is fold-like if and only if

$$X_p \quad \text{and} \quad (\eta_\varphi)_p$$

are linearly independent, where η_φ is a null section of φ.

We have the following corollary.

Corollary 4.4 ([9]). If $p \in S$ is a cusp-like singular point, then $X_p \notin T_pS_2$. If $p \in S$ is a swallowtail-like singular point. Then

$$\tilde{\mu}(0) = 0, \quad \tilde{\mu}'(0) \neq 0,$$

where $\gamma(t) = (\gamma_1(t), \gamma_2(t))$ $(\gamma(0) = p)$ is a parameterization of S_2, and $\eta_{\gamma(t)} = a(t)\partial_u + b(t)\partial_v$, and

$$\tilde{\mu}(t) = \begin{pmatrix} \gamma_1(t) & a(t) \\ \gamma_2(t) & b(t) \end{pmatrix}.$$

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