Theory of solvability of generalized Hamiltonian systems and a study on abnormal extremals of rank two distributions

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- A study of abnormal extremals on a sub-Riemannian manifold

Theorem (T)

Let $(M, \mathcal{D}, g)$ be a sub-Riemannian smooth manifold with a distribution $\mathcal{D}$ of rank two. Suppose that $\mathcal{D}_1 := \mathcal{D} + [\mathcal{D}, \mathcal{D}]$ is a sub-bundle of rank three and $\mathcal{D}_2 := \mathcal{D}_1 + [\mathcal{D}, \mathcal{D}_1]$ is a sub-bundle of rank four. Then for any point q_0 in M, there exist a closed submanifold V_{q_0} of q_0 in M and a smooth $(2n - 4)$ parameter family of C^∞ immersive abnormal bi-extremal $\{\gamma_q(t)\}_{q \in V_{q_0}}$ of which projection are not normal geodesics defined on a small interval.

Motivation (Sub-Riemannian geometry)

Definition

- M: C^∞ manifold
- $\mathcal{D}$: a distribution (i.e. a sub-bdl. of TM)
- g : a bi-linear positive definite form on $\mathcal{D}$

$\longrightarrow$ $(M, \mathcal{D}, g)$: sub-Riemannian manifold.

- A horizontal curve : abs. conti. curve $\gamma\colon I \to M$ s.t. $\dot{\gamma}(t)$ is measurable & bounded & $\dot{\gamma}(t) \in \mathcal{D}_{\gamma(t)}$ a.e. $t \in I$.

Chow–Rashevsky's theorem

If $\mathcal{D}$ satisfies Hörmander's condition on conn. mfd M, every two points are connected by a horizontal curve.

$\longrightarrow$ we may define Carnot–Carathéodory (or sub-Riemannian) distance.

$$d_{CC}(p,q) := \inf_{\gamma}\Big\{L(\gamma) := \int_{[a,b]} \sqrt{g(\dot{\gamma}(t), \dot{\gamma}(t))}dt \mid \gamma\colon [a,b] \to M : horizontal, \gamma(a) = p, \gamma(b) = q\Big\}$$

- A minimizer γ: a horizontal curve γ connecting p and q s.t. $d_{CC}(p,q) = L(\gamma)$.

Extremals

Local minimizers $\longrightarrow$ Extremals

Normal extremal

$H_E \in C^\infty(T^*M, \mathbb{R})$

$H_E(x,p) = -\frac{1}{2}\sum_{i,j} g^{ij}(x)p_i p_j,$

$$\dot{x}(t) = \frac{\partial H_E}{\partial p}(x(t), p(t))$$
$$\dot{p}(t) = -\frac{\partial H_E}{\partial x}(x(t), p(t))$$

Abnormal extremal

$H\colon T^*M \times_M \mathcal{D} \to \mathbb{R}$

$H(x,p,u) := \langle p_x, u \rangle,$

$$\dot{x}(t) = \frac{\partial H}{\partial p}(x(t), p(t), u(t)),$$
$$\dot{p}(t) = -\frac{\partial H}{\partial x}(x(t), p(t), u(t)),$$
$$\frac{\partial H}{\partial u_i}(x(t), p(t), u(t)) = 0 \quad (1 \le i \le k).$$

Extremals

Let $\{X_1, \dots, X_k\}$ be a local framing of $\mathcal{D}$.

Normal extremals (normal geodesics)

Set C^∞ function $H_E \colon T^*M \to \mathbb{R}$ as

$$H_E(x,p) = -\frac{1}{2}\sum_{i,j} g^{ij}(x)\langle p, X_i(x)\rangle\langle p, X_j(x)\rangle$$

where $g_{ij} = g(X_i, X_j)$ and $(g^{ij})_{i,j}$ is the inverse matrix of $(g_{ij})_{i,j}$.
A solution of the Hamiltonian equation associated to H_E is called a normal bi-extremal and its projection to M is called a normal extremal (or a normal geodesic). The eqn's are expressed as

$$\dot{x}(t) = \frac{\partial H_E}{\partial p}(x(t), p(t)), \quad \dot{p}(t) = -\frac{\partial H_E}{\partial x}(x(t), p(t))$$

with Darboux coordinates (x,p) of T^*M.

Extremals

Set

- $H\colon T^*M \times_M \mathcal{D} \to \mathbb{R}$, $H(x,p,u) := \langle p, u\rangle$ for $x \in M, p \in T_x^*M$ and $u \in \mathcal{D}_x$.

Abnormal extremals

The constrained Hamiltonian system for H is def. by

$$\begin{cases} \dot{x}(t) = \dfrac{\partial H}{\partial p}(x(t), p(t), u(t)), \\ \dot{p}(t) = -\dfrac{\partial H}{\partial x}(x(t), p(t), u(t)), \\ \dfrac{\partial H}{\partial u_i}(x(t), p(t), u(t)) = 0 \ (1 \leq i \leq k). \end{cases}$$

If $\exists x(t)$: a horizontal curve on $[0, \varepsilon)$, $p(t)$ on $T^*_{x(t)}M \setminus \{0\}$ and $u(t) \in \mathcal{D}_{x(t)}$ which satisfies the equations for a.e. $t \in [0, \varepsilon)$, the curve $(x(t), p(t))$ on T^*M is called an abnormal bi-extremal and the projection of an abnormal bi-extremal to M is called an abnormal extremal.

- Local minimizers $\longrightarrow$ Extremals

- Extremals $\longrightarrow$ local minimizers?

- Local minimizers $\longrightarrow$ Extremals
- Extremals $\longrightarrow$ local minimizers? No. (R. Montrgomery)

But not much is known about abnormal extremals.

Abnormal extremal

$\boldsymbol{H : T^*M \times_M \mathcal{D} \to \mathbb{R}}$

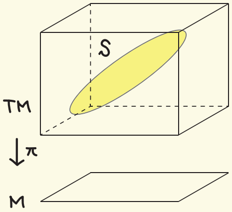
$\boldsymbol{H(x, p, u) := \langle p_x, u \rangle}$,

$$\dot{x}(t) = \frac{\partial H}{\partial p}(x(t), p(t), u(t)),$$
$$\dot{p}(t) = -\frac{\partial H}{\partial x}(x(t), p(t), u(t)),$$
$$\frac{\partial H}{\partial u_i}(x(t), p(t), u(t)) = 0 \quad (1 \leq i \leq k).$$

$\longleftarrow$ Study the constrained system as an implicit differential system

Implicit differential systems (cf. Fukuda–Janeczko 2004)

- M: m-dim. C^∞ manifold.
- $\pi : TM \to M$: tangent bundle projection



Definition

An implicit differential system on M is a submanifold $S \subset TM$.

Example

For a vector field $X \in \Gamma(TM)$, set

$$gragh(X) := \{(x, \dot{x}) \in TM \mid \dot{x} = X(x)\}.$$

$gragh(X)$ is an implicit differential system.

Several natural questions from theory of ODEs.

- existence of a (local) solution
- uniqueness of the solution of Cauchy problem
- smoothness of solutions
- a range of the largest domain of prolongation of solutions

and so on...

$\longrightarrow$ We stick to consider the first two problems.

Smooth solvability over submanifolds

Let N be a submanifold of M.

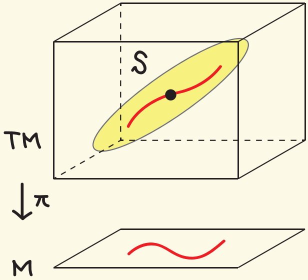
Definition

- A solution of S over N is a C^1 curve $\gamma\colon (a,b) \to N$ s.t. $(\gamma(t), \dot\gamma(t)) \in S \cap \pi^{-1}(N)$ for all $t \in (a,b)$.
- A point $(x_0, \dot x_0) \in S$ is a solvable point of S over N if $\exists \varepsilon > 0$ and $\exists \gamma\colon (-\varepsilon, \varepsilon) \to N$: solution s. t. $(\gamma(0), \dot\gamma(0)) = (x_0, \dot x_0)$.
- A point $(x_0, \dot x_0) \in S$ is a smoothly solvable point of S over N if $\exists W \subset S \times \mathbb{R}$: open nbd. of $(x_0, \dot x_0, 0)$, $\exists \bar\gamma\colon W \to N$:$C^\infty$ map s.t.

  $$\gamma_{(x,\dot x)}(t) := \bar\gamma(x, \dot x, t)$$

  is a solution of S over N with $(\gamma(0), \dot\gamma(0)) = (x, \dot x)$ $\forall (x, \dot x) \in \pi_1(W)$, where $\pi_1\colon S \times \mathbb{R} \to S$ is a natural projection.
- An implicit differential system S over N is called a smoothly solvable submanifold over N if S consists only of smoothly solvable points of S over N.

Smooth solvability over submanifolds



Necessary condition of smooth solvability.

Tangential solvability condition

A point $(x, \dot{x}) \in S$ is solvable point. Then

$$\dot{x} \in d(\pi|_S)_{(x,\dot{x})}(T_{(x,\dot{x})}S).$$

QUESTION: What is a sufficient condition for S to be smoothly solvable? There are some answers (cf. Fukuda-Janeczko, 2004).

We consider such conditions for symplectic case.

Implicit Hamiltonian systems

- (M, ω) a symplectic manifold.

$\longrightarrow$ Tangent bundle $(TM, \dot{\omega})$: a symplectic manifold.
where

- $\flat : TM \to T^*M$: bdl. isom. def. by $\flat_x(v_q) = \iota_{v_q}\omega_q$, $q \in M$,
- θ : the Liouville form on T^*M,
- $\dot{\omega} := \flat^* d\theta$.

In what follows we set $M = \mathbb{R}^{2n}$ with the standard symplectic form ω.

Definition (Fukuda–Janeczko, 2004)

A Lagrangian submanifold L of $(T\mathbb{R}^{2n}, \dot{\omega})$ (i.e., $\dim L = 2n$ and $\dot{\omega}|_L = 0$) is called an implicit Hamiltonian system.

Generalized Hamiltonian systems

Now we focus on implicit Hamiltonian systems which is generated by a Morse family of particular type:

$$F\colon \mathbb{R}^{2n} \times \mathbb{R}^k \to \mathbb{R}, \quad F(x,p,u) = \sum_{j=1}^{k} a_j(x,p)u_j + b(x,p).$$

The catastrophe set of F:

$$\begin{aligned} C(F) &= \left\{ (x,p,u) \in \mathbb{R}^{2n} \times \mathbb{R}^k \mid \frac{\partial F}{\partial u_i}(x,p,u) = 0, i = 1,\dots,k \right\} \\ &= K \times \mathbb{R}^k \end{aligned}$$

where $K := \{(x,p) \in \mathbb{R}^{2n} \mid a_i(x,p) = 0, i = 1,\dots,k\}$.
$\phi_F\colon \mathbb{R}^{2n} \times \mathbb{R}^k \to T\mathbb{R}^{2n}$

$$\phi_F(x,p,u) = (x, p, \frac{\partial F}{\partial p_i}(x,p,u), -\frac{\partial F}{\partial x_i}(x,p,u)).$$

$\longrightarrow L_F = \phi_F(C(F))$: a generalized Hamiltonian system.

Smoothly solvable submanifolds of L_F over submanifolds

The case $b \equiv 0$.
L_F induced from co-normal bundles of the submanifold K.

For the case $k = 2$.
- $F\colon \mathbb{R}^{2n} \times \mathbb{R}^2 \to \mathbb{R}$: Morse family def. by

$$F(x,p,u) = a_1(x,p)u_1 + a_2(x,p)u_2.$$

Notation

- $\langle a_1, a_2, \{a_1, a_2\}\rangle_{\mathcal{E}_{\mathbb{R}^{2n},q_0}}$: $\mathcal{E}_{\mathbb{R}^{2n},q_0}$-module gen. by a_1, a_2 and $\{a_1, a_2\}$.
- $\xi_1 := \{a_1, \{a_1, a_2\}\}$, $\xi_2 := \{a_2, \{a_1, a_2\}\}$
- $A_1 := \{(x,p) \in K \mid a_1 = a_2 = \{a_1, a_2\} = 0\}$

Proposition (T)

Assume that a_1, a_2, and $\{a_1, a_2\}$ are independent. Then $\phi_F(A_1 \times \mathbb{R}^2)$ is a smoothly solvable submanifold of L_F over A_1 if and only if $\xi_1, \xi_2 \in \langle a_1, a_2, \{a_1, a_2\}\rangle_{\mathcal{E}_{\mathbb{R}^{2n},q_0}}$ for any point q_0 in A_1.

Proposition (T)

Assume that a_1, a_2 and $\{a_1, a_2\}$ are independent. Assume also that

$$\xi_2 \in \langle a_1, a_2, \{a_1, a_2\}\rangle_{\mathcal{E}_{\mathbb{R}^{2n},q_0}} \text{ and } \xi_1 \notin \langle a_1, a_2, \{a_1, a_2\}\rangle_{\mathcal{E}_{\mathbb{R}^{2n},q_0}}$$

at every point q_0 of A_1. Then the followings hold.

1. $\phi_F(\overline{A_{1,1}}^2)$ is a smoothly solvable submanifold of L_F over $A_{1,1}$.
2. Assume, furthermore, that $\xi_1, a_1, a_2, \{a_1, a_2\}$ are independent.
    1. $\phi_F(\overline{A^1_2}^2)$ is a smoothly solvable submanifold of L_F over A^1_2.
    2. $\phi_F(A^1_2 \times \mathbb{R}^2)$ is a smoothly solvable submanifold of L_F over A^1_2 if $\{a_1, \xi_1\} \in \langle a_1, a_2, \{a_1, a_2\}, \xi_1\rangle_{\mathcal{E}_{\mathbb{R}^{2n},q_0}}$ for any point q_0 in A^1_2.

Proposition (T)

Assume that a_1, a_2 and $\{a_1, a_2\}$ are independent. Assume also that

$$\xi_2 \in \langle a_1, a_2, \{a_1, a_2\}\rangle_{\mathcal{E}_{\mathbb{R}^{2n},q_0}} \text{and } \xi_1 \notin \langle a_1, a_2, \{a_1, a_2\}\rangle_{\mathcal{E}_{\mathbb{R}^{2n},q_0}}$$

at every point q_0 of A_1. Then the followings hold.

1. $\phi_F(\overline{A_{1,1}}^2)$ is a smoothly solvable submanifold of L_F over $A_{1,1}$.
2. Assume, furthermore, that $\xi_1, a_1, a_2, \{a_1, a_2\}$ are independent.
    1. $\phi_F(\overline{A_2^1}^2)$ is a smoothly solvable submanifold of L_F over A_2^1.
    2. $\phi_F(A_2^1 \times \mathbb{R}^2)$ is a smoothly solvable submanifold of L_F over A_2^1 if $\{a_1, \xi_1\} \in \langle a_1, a_2, \{a_1, a_2\},$

Proposition (T)

Assume that a_1, a_2 and $\{a_1, a_2\}$ are independent. Assume also that

$$\xi_1 \in \langle a_1, a_2, \{a_1, a_2\}\rangle_{\mathcal{E}_{\mathbb{R}^{2n},q_0}} \text{and } \xi_2 \notin \langle a_1, a_2, \{a_1, a_2\}\rangle_{\mathcal{E}_{\mathbb{R}^{2n},q_0}}$$

at every point q_0 of A_1. Then the followings hold.

1. $\phi_F(\overline{A_{1,2}}^1)$ is a smoothly solvable submanifold of L_F over $A_{1,2}$.
2. Assume, furthermore, that $\xi_2, a_1, a_2, \{a_1, a_2\}$ are independent.
    1. $\phi_F(\overline{A_2^2}^1)$ is a smoothly solvable submanifold of L_F over A_2^2.
    2. $\phi_F(A_2^2 \times \mathbb{R}^2)$ is a smoothly solvable submanifold of L_F over A_2^2 if $\xi_2\rangle_{\mathcal{E}_{\mathbb{R}^{2n},q_0}}$ for any point q_0 in A_2^2.

In the case $\xi_1, \xi_2 \notin \langle a_1, a_2, \{a_1, a_2\}\rangle_{\mathcal{E}_{\mathbb{R}^{2n},q_0}}$ we have

Proposition (T)

Assume that a_1, a_2 and $\{a_1, a_2\}$ are independent. Then $\phi_F(\overline{A_{1,(1,2)}}^{1,2})$ is a smoothly solvable submanifold of L_F over $A_{1,(1,2)}$ if $\xi_1, \xi_2 \notin \langle a_1, a_2, \{a_1, a_2\}\rangle_{\mathcal{E}_{\mathbb{R}^{2n},q_0}}$ for every point q_0 in $A_{1,(1,2)}$.

Proposition (T)

Assume that a_1, a_2, $\{a_1, a_2\}$ and ξ_1 are independent. Then $\phi_F(\overline{A_2^1}^1)$ is a smoothly solvable submanifold of L_F over A_2^1 if $\{a_1, \xi_1\} \in \langle a_1, a_2, \{a_1, a_2\}, \xi_1\rangle_{\mathcal{E}_{\mathbb{R}^{2n},q_0}}$ for any point q_0 in A_2^1.

Proposition (T)

Assume that a_1, a_2, $\{a_1, a_2\}$ and ξ_2 are independent. Then $\phi_F(\overline{A_2^2}^2)$ is a smoothly solvable submanifold of L_F over A_2^2 if $\{a_2, \xi_2\} \in \langle a_1, a_2, \{a_1, a_2\}, \xi_2\rangle_{\mathcal{E}_{\mathbb{R}^{2n},q_0}}$ for any point q_0 in A_2^2.

A lemma for the proof

Let
- A : a submanifold of K, C : a submanifold of $\mathbb{R}^k$
- $\widetilde{A}$: a submanifold of $A \times \mathbb{R}^k$
- $\alpha \colon \widetilde{A} \to A$ a locally trivial fibration which fibre is C with $\alpha(q, v) := q$ for $(q, v) \in \widetilde{A} \subset A \times \mathbb{R}^k$.
- $X_u \colon A \to T(\mathbb{R}^{2n})$: a family of vector fields along A

$$X_u(x, p) = (x, p, \frac{\partial F}{\partial p}(x, p, u(x, p)), -\frac{\partial F}{\partial x}(x, p, u(x, p)).$$

Lemma (T)

A point $(q_0, \dot{q}_0) \in \phi_F(\widetilde{A})$ is a smoothly solvable point of $\phi_F(\widetilde{A})$ over A if there exist an open neighborhood V_0 of q_0 in A and a smooth map $s \colon V_0 \times C \to \widetilde{A}$ of smooth family of sections

$$s_c := s(\cdot, c) \colon V_0 \to \widetilde{A}$$

for each $c \in C$ such that for any $(q, \dot{q}) \in \phi_F(\alpha^{-1}(V_0))$ there exists $c \in C$ which satisfies

$$\phi_F(s_c(q)) = \dot{q}$$

and X_{S_c} is tangent vector field on V_0.

An application to sub-Riemannian geometry

Lie algebra homomorphism

$$(\Gamma(TM), [\cdot,\cdot]) \to (C^\infty(T^*M,\mathbb{R}), \{\cdot,\cdot\}), \quad X \mapsto \langle p, X(x)\rangle$$

Proposition (T)

For a rank 2 distribution $\mathcal{D}$ with small growth vector $(2,3,3,\ldots)$ at each point in M, there exist a close manifold S of $T^\sharp M$ with codimension 3 and a smooth $(2n-3)$-parameter family of totally singular abnormal bi-extremals $\{(x_q(t), p_q(t))\}_{q\in S}$ in S.

Proposition (T)

For a rank 2 distribution $\mathcal{D}$ with small growth vector $(2,3,4,\ldots)$ at each point in M, there exist a close submanifold S of $T^\sharp M$ and a smooth family of abnormal bi-extremals $\{(x_q(t), p_q(t))\}_{q\in S}$ in S which is either regular or totally singular.

Here $(x(t), p(t))$ is

$$\text{regular} \longleftrightarrow p(t) \in (\mathcal{D})^\perp_{x(t)} \setminus ([\mathcal{D},\mathcal{D}])^\perp_{x(t)}$$

$$\text{totally singular} \longleftrightarrow p(t) \in ([\mathcal{D},\mathcal{D}])^\perp_{x(t)}$$

Theorem (T)

Let $(M, \mathcal{D}, g)$ be a sub-Riemannian smooth manifold with a distribution $\mathcal{D}$ of rank two. Suppose that $\mathcal{D}_1 := \mathcal{D} + [\mathcal{D}, \mathcal{D}]$ is a sub-bundle of rank three and $\mathcal{D}_2 := \mathcal{D}_1 + [\mathcal{D}, \mathcal{D}_1]$ is a sub-bundle of rank four. Then for any point q_0 in M, there exist a closed submanifold V_{q_0} of q_0 in M and a smooth $(2n-4)$ parameter family of C^∞ immersive abnormal bi-extremal $\{\gamma_q(t)\}_{q \in V_{q_0}}$ of which projection are not normal geodesics defined on a small interval.

$$
\begin{array}{ccc}
(x(t), \dot{x}(t)) \in \mathcal{D} & \longleftarrow - - - - - & (x(t), p(t), \dot{x}(t), \dot{p}(t)) \in L_F \\
\Big\downarrow \pi_{TM} & \circlearrowleft & \Big\downarrow \pi_{T(T^*M)} \\
x(t) \in M & \xleftarrow[\pi_{T^*M}]{} & (x(t), p(t)) \in A \subset T^*M
\end{array}
$$

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Thank you for your attention !!

Supplementation

Hörmander condition

$\mathcal{D}$ satisfies Hörmander condition if $\exists d \in \mathbb{N}$ s. t. $\forall x \in M$, a local framing $\{X_1, \ldots, X_k\}$ of $\mathcal{D}$ around x satisfies

$$span\{X_1, \ldots, X_k, [X_i, X_j], \ldots, [X_{i_1}, [X_{i_2}, [\cdots, [X_{i_{d-1}}, X_{i_d}], \cdots,]]]\} = T_x M$$

For a bounded measurable curve $c\colon [0, T] \to \mathcal{D}$, if a curve $\gamma := \pi_{\mathcal{D}} \circ c\colon [0, T] \to M$ satisfies $\dot{\gamma}(t) = c(t)$ for almost everywhere on $[0, T]$, then γ is a horizontal curve and c is called an admissible velocity. Here $\pi_{\mathcal{D}}\colon \mathcal{D} \to M$ is the canonical projection.

Endpoint mapping

The map

$$\mathbf{End}(q_0)\colon \mathcal{V}_{q_0} \to M, c \mapsto \gamma(T)$$

is called an *end-point mapping* and is differentiable by means of Fréchet derivative.