Vertex Algebras Associated to Toroidal Algebras

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Outline

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- Toroidal Algebras
- Goal: Reconstruct Toroidal Vertex Algebras

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  - open $U \mapsto \mathcal{F}(U) \in \mathcal{C}$
  - inclusion $U \hookrightarrow V$, $\mathcal{F}(U) \to \mathcal{F}(V)$
  - disjoint $U_1, \ldots, U_k \subset V$, $\mathcal{F}(U_1) \otimes \ldots \otimes \mathcal{F}(U_k) \to \mathcal{F}(V)$

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  - disjoint $U_1, \ldots, U_k \subset V$, $\mathcal{F}(U_1) \otimes \ldots \otimes \mathcal{F}(U_k) \to \mathcal{F}(V)$
- $\mathcal{F}$ satisfies natural coherence and gluing conditions.

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- If $\mathcal{F}$ is translation invariant and carries an S^1 action (in a precise sense, see [Costello-Gwilliam, 2016]), one can recover a vertex algebra.

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- $[\cdot, K] = 0$, i.e. K is central.
- $[A \otimes f(t), B \otimes g(t)] = [A, B] \otimes f(t)g(t) - \mathrm{Res}_{t=0} f dg (A, B) K$ where $(\cdot, \cdot) = \frac{1}{2h^\vee}(\cdot, \cdot)_K$.

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  of level k where $\mathbb{C}_k$ is 1-dim. rep'n of $L\mathfrak{g}_+ \oplus \mathbb{C}K \subset \hat{\mathfrak{g}}$.
- $L\mathfrak{g}_+ = \mathfrak{g} \otimes \mathbb{C}[t]$ acts by zero and K acts by k.

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- Obtain factorization algebra by taking Chevalley-Eilenberg cochains,

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- The vertex algebra recovered from $\mathcal{F}_\kappa$ is isomorphic to the vertex algebra $V_\kappa(\mathfrak{g})$.

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- For $N = 1$, this is just the affine Kac-Moody algebra. For $N > 1$, the central term is infinite dimensional.
- There have been vertex algebras associated to representations of this Lie algebra [Berman-Billig-Szmigielski, 2013].

Goal: Recover toroidal vertex algebras via factorization algebras

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- $\overline{\Omega}(U) = \overline{\Omega}^{0,*}(U \times X, \overline{\partial}_{t_0} + \overline{\partial}_X) =$ smooth forms of type $(0, *)$ which are zero outside $K \times X$ where $K \subset\subset U$.

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- $\mathcal{F}(U) = C_*(\overline{\Omega}(U))$ defines a factorization algebra on $\mathbb{C}$.

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- Idea: Relate this vertex algebra to vertex algebras associated to toroidal algebras as found in literature.

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