Picard Lefschetz theory on symplectic orbifolds

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① Lefschetz fibration and related geometry

Let's start from a holomorphic function $\pi : X^n \to \mathbb{C}$ from an affine mfd. f can be perturbed into non-degenerate. i.e. locally around $df(x_0) = 0$. looks like: $\pi = z_1^2 + z_2^2 + \cdots + z_n^2$ in coordinate $(z_1, \cdots, z_n)$. $\Rightarrow$ Lefschetz fibration

ex 1:
More familiar if Y is projective $\Rightarrow$ hyperplane sections gives a Lefschetz pencil.

This is a compactification of Lef. fibration

exer: On $\mathbb{P}^2$, (i) lines through a point
Remove one line $\Rightarrow$ trivial $\mathbb{C}$-fib. over $\mathbb{C}$
oral

quadric through 4 pts.

$$Z_t = \{ f + tg = 0 \}, \quad t \in \mathbb{P}^1 \setminus \{\infty\}$$

quadratic

fibration of $\mathbb{P}^2 \setminus Z_\infty$. C.P. $= \{ g = 0 \}$

ex 2:
For symplectic geometry:

(M^{2n}, ω), ω^n = volume form.

All affines are symplectic. ω_{std} restriction. Kähler forms.

Non-algebraic examples that we care a lot:

$\forall N$ smooth, $(T^*N, d\lambda)$ is symplectic. $\lambda = pdq$
(classical mechanics) (p_i, q_i) coord.

Interesting class of Submfd: Lagrangians.

L^n s.t. $\omega|_L = 0$. if $\lambda|_L$ is also exact, L is called exact Lag.

Uncertainty principle: "Lag. is minimal meaningful geometry."

$N \subset T^*N$ position. $T^*_x N$ momentum.

A central problem (Arnold):

All Lag. in T^*N are isotopic to zero section.

Lemma: T^*N has a Lefschetz fibration:

Nash–Tognoli: $N^n \hookrightarrow \mathbb{R}^{2n+1}$, C^2 approximate by real variety. $U_{\mathbb{R}}$.

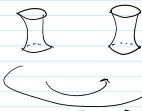
Complexify $\Rightarrow U \subset \mathbb{C}^{2n+1}$ (Need to deal w/ singularities)

But $nbh(U_{\mathbb{R}})$ is like $T^*U_{\mathbb{R}} = T^*N$.

Use compactification to $\mathbb{C}P^{2n+1}$ + Hironaka's resolution @ ∞, then use Lefschetz hyperplane sections therein.

② Local symp. geom in Lef. fib from singularity thy

$$(z_1, \cdots, z_n) \mapsto \sum z_i^2$$



T^*S^n

monodromy = DT along vanishing cycle.

$\Rightarrow \tau_{S^n}$ cpt supp. diffeomorphism s.t. $\tau_{S^n}^* \omega = \omega$

When $n=1$, it's actual DT.

③ What makes this more interesting for symp. geometers: Donaldson

HMS: (Kontsevich) $\underbrace{D^\pi Fuk(X)}_{\text{Some cat from Lag. sub mfd.}} \cong \underbrace{D^b(\check{X})}_{\text{derived cat of coherent sheaves}}$

Classical problems in symplectic topology.

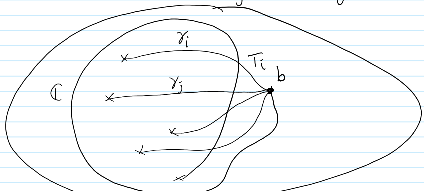
Increasing interests. complete init for Fano/anti Fano

$X, \check{X}$ are originally set-up as CY's. As people explored extensions for this conjecture combining geometry and physics. $X, \check{X}$ are extended to much more general situations.

One instance: X = LG-model
$\check{X}$ = Fano varieties ← write this first.

LG-model: $\pi : X \to \mathbb{C}$. After perturbation, C.P. of π becomes non-degenerate $\Rightarrow$ Lefschetz fibration!

$D^\pi Fuk(\pi)$ is defined by the following data:



Parallel transport of fibers over γ_i will incur a Lag. D^{n+1} over each fiber $= S^n$ (Lefschetz thimbles T_i)

(4) Fiber cohomology, Fukaya category and triangulated structures

Given 2 Lag. submfds in X L_0, L_1 $\Rightarrow$ (graded) chain cxes over $\mathbb{Z}_2$
exact, possibly immersed; with appropriate tech. restriction
$CF^*(L_0, L_1) \Rightarrow HF^*(L_0, L_1)$

Take a category with:
Objects: T_i + exact Lag.
Morphisms: $CF^*(L_0, L_1)$
with additional algebraic structures (A_∞ structures)

Thm: (Seidel, Biran-Cornea) T_i generates $Fuk(\pi)$.
Roughly, "generation" means one may recover any objects from T_i by mapping cones:

basic operation:

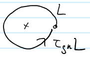
$$\begin{array}{ccc} L_0 & \to & L_1 \\ & {\scriptstyle [1]}\nwarrow & \downarrow \\ & & L_2 \end{array}$$

is triangle $\Rightarrow$ $\exists$ LES. $HF(N, L_0) \to HF(N, L_1) \to HF(N, L_2)$ $[1]$

need efforts.

Heart of Seidel's proof:
$CF(S^n, L) \otimes S^n \to L \to \tau_{S^n} L$ $[1]$ is a Cone

effect of monodromy



An application (Fukaya-Seidel-Smith, Abouzaid)
$L \subset T^*N$ exact, then its projection to N is a homotopy equivalence. (Represent all other Lag. as modules)

(5) Recent developement of Picard-Lefschetz thy.

The compactification of D provides an algebraic deformation for $Fuk(\pi)$ $\rightsquigarrow$ Connects to $Fuk(M)$.

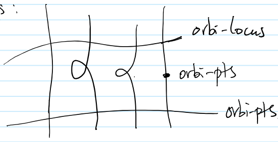
More interestingly, this deformation is governed by GW invts of M. (Seidel)

(6) Symplectic orbifolds: locally like $\mathbb{R}^{2n}/\Gamma$ — finite gp. Isotropy

Work in progress (w/ GB. Xu)
For any closed symp. orbifolds M, $\exists$ symp. embedding with cyclic isotropies into wt. projective spaces. Hyperplane sections $\Rightarrow$ Lefschetz fibrations.

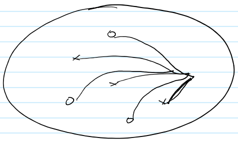
For non-cyclic ones, the embedding goes to wt. Grassmannians

orbi-Lefschetz fibrations:



Unless the obi-point is isolated, the orbifold locus ⋔ fibration.
Locally near an orbi-pt, $\Gamma = \mathbb{Z}/N$, $\pi = f(z)$ st. $f(z) = f(g \cdot z)$
on chart.

$\Rightarrow$



Want a Picard-Lef. thy.

⑦ Local analysis of orbifold pts.

$$\begin{array}{ccc} \mathbb{C}^k/(\mathbb{Z}/N) & \xleftarrow{\ p\ } & \mathbb{C}^k \\ \downarrow \pi & & \downarrow \tilde{\pi} \\ \mathbb{C} & & \mathbb{C} \end{array} \quad , \ \tilde{\pi} \text{ is } \mathbb{Z}/N\text{-invariant}$$

* Assume p does not have a fiberwise component. (Isotropy of a fiber = id)

Approach 1: Perturb $\tilde{\pi}$ $\Rightarrow$ $\mathrm{Fuk}(\tilde{\pi})$ and find group actions on it.

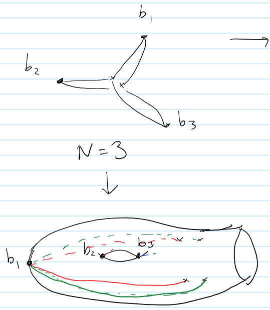
(hard to pin down which, not trivial to find)

Approach 2: Regard $\mathrm{Fuk}(\pi)$ as $\mathbb{Z}/2$-eq. Fuk of double cover.

i) consider $p^{-1}(F_0)$ in $\mathbb{C}^k$
$\uparrow$ general fiber

ii) Consider $(\mathbb{Z}/N \rtimes \mathbb{Z}/2)$-eq. Fuk cat on double cover over $\mathbb{C}^k$ branched along $p^{-1}(F_0)$

$\longrightarrow$ this is a Lef. fib. over a general Riemann Surface with a natural $\mathbb{Z}/N \rtimes \mathbb{Z}/2$-action



Finally, we want to package into directed Fuk cat on fiber. This will give us a $\mathbb{Z}/N$-eq. cat made out of N copies of each vanishing cycle.

Question: How to put this into a global Lefschetz fibration picture?

* When $\exists$ isotropy Γ of fibers.

$$\{ z_1^2 + z_2^2 + \cdots + z_{2k-1}^2 + z_{2k}^2 = 1 \}$$

c.o.v.
$$\Leftrightarrow \{ x_1 y_1 + \cdots + x_k y_k = 1 \}$$

$$\zeta_m \cdot (x_i, y_i) = (\zeta_m x_i, \zeta_m^{-1} y_i),$$

$\Rightarrow$ Vanishing cycle = Lens spaces S^n/Γ, but the actual object that counts is the immersed sphere $S^n \to S^n/\Gamma$

Thm (Mak-W.) effect of monodromy is

$$CF^*(S^n, L) \otimes_\Gamma S^n \longrightarrow L \to \tau_{S^n/\Gamma} L$$
$[1]$

Expectation: $\mathrm{Fuk}(\pi)$ = block-directed.
blocks = spheres / lens spaces
AND those from local orbifold models.

Further direction: Consider deformation thy by compactifications.
Recover orbifold GW. from these.
Seidel's ODE's.