SYZ mirror symmetry of hypertoric varieties

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Outline

- What is mirror symmetry?
- What are hypertoric varieties?
- SYZ mirror construction for hypertoric varieties.

Section 1

What is mirror symmetry?

Mirror Symmetry

- Mirror symmetry was discovered in the early 90's by **Greene-Plesser** and **Candelas-De la Ossa-Green-Parkes**.
- It asserts that Calabi-Yau manifolds come in mirror pairs $(X, \check{X})$, with dualities:

$$\text{Symplectic geometry}(X) \longleftrightarrow \text{Complex geometry}(\check{X}),$$
$$\text{Complex geometry}(X) \longleftrightarrow \text{Symplectic geometry}(\check{X}).$$

- $\exists$ deep relations between X and $\check{X}$:
    1. symmetry of Hodge diamonds $h^{p,q}(X) = h^{n-p,q}(\check{X})$,
    2. counting of rational curves in X and period integral in $\check{X}$,
    3. equivalence of derived categories $DFuk(X) \cong D^b Coh(\check{X})$
    4. etc...

The SYZ Conjecture

- Fundamental question: Given a Calabi-Yau manifold X, how to construct its mirror $\check{X}$ geometrically?
- In 1996, Strominger-Yau-Zaslow proposed that mirror symmetry is T-duality.

Conjecture

X and $\check{X}$ admit **dual Lagrangian torus fibrations** $\mu : X \to B$ *and* $\check{\mu} : \check{X} \to B$ *over the same base B. Namely for a regular value $b \in B$, $\mu^{-1}(b)$ and $\check{\mu}^{-1}(b)$ are dual tori.*

- Thus, given a Lagrangian torus fibration $\mu : X \to B$, $\check{X}$ can be reconstructed as the total space of dual tori $\check{\mu}^{-1}(b) = \mathrm{Hom}(\pi_1(\mu^{-1}(b)), U(1))$.

Section 2

What are hypertoric varieties?

Constructing hypertoric varieties

- Hypertoric varieties: hyperkhler analogue of toric varieties.
- Construction: hyperkhler quotient or GIT quotient of $T^*\mathbb{C}^n$.
- $\vec{t} \in K \subset T^n$ acts on $(T^*\mathbb{C}^n, dz_i \wedge d\bar{z}_i, dz_i \wedge dw_i)$ by $\vec{t} \cdot (\vec{z}, \vec{w}) = (t_i z_i, t_i^{-1} w_i)$, and gives moment maps

$$\begin{array}{ccc} T^*\mathbb{C}^n & \xrightarrow{(|z_i|^2 - |w_i|^2, z_i w_i)} & (\mathfrak{t}^n)^* \oplus (\mathfrak{t}^n_{\mathbb{C}})^* \\ & \searrow^{(\mu_{\mathbb{R}}, \mu_{\mathbb{C}})} & \downarrow \iota^* \\ & & (\mathfrak{t}^k)^* \oplus (\mathfrak{t}^k_{\mathbb{C}})^* \end{array}$$

- Choose $(\theta, \lambda) \in (\mathfrak{t}^k)^* \oplus (\mathfrak{t}^k_{\mathbb{C}})^*$, the hyperkhler quotient

$$X_{\theta,\lambda} = (\mu_{\mathbb{R}}, \mu_{\mathbb{C}})^{-1}(\theta, \lambda)/K$$

is called a hypertoric variety.
- Alternatively, one can construct $X_{\theta,\lambda}$ as the GIT quotient

$$X_{\theta,\lambda} = \mu_{\mathbb{C}}^{-1}(\lambda)//_{\theta} K_{\mathbb{C}}$$

where $\theta : K \to C^{\times}$ is the stability parameter, and $K_{\mathbb{C}}$ is K complexified.
- $X_{\theta,\lambda}$ is Calabi-Yau since $\mathrm{Hol}(X_{\theta,\lambda}) \subset Sp(d) \subset SU(2d)$, $d = n - k$.
- Examples: $T^*\mathbb{P}^n$, $\widetilde{A_n}$ the crepant resolution of A_n singularities, etc...

Hyperplane Arrangements

- Toric varieties $\longleftrightarrow$ polytopes.

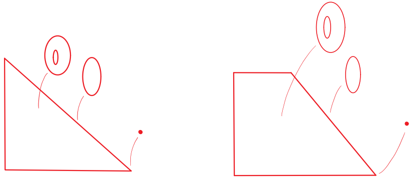
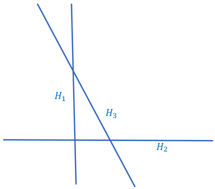


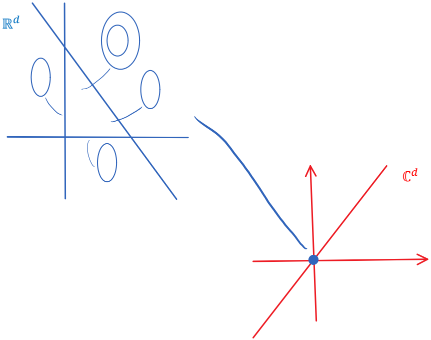
Figure: moment polytopes of $\mathbb{P}^2$ and $\mathcal{H}_2$

- Hypertoric varieties $\longleftrightarrow$ hyperplane arrangements $\{H_i\}_{i=1}^n$.
- Quotient torus $T^n/K = T^d$ acts on $X_{\theta,\lambda}$, and gives moment maps $(\bar{\mu}_{\mathbb{R}}, \bar{\mu}_{\mathbb{C}}) : X_{\theta,\lambda} \twoheadrightarrow \mathbb{R}^d \oplus \mathbb{C}^d$. We have hyperplane arrangements in both $\mathbb{R}^d$, and $\mathbb{C}^d$.

- Example: hyperplane arrangements for $\widetilde{A_n} = T^*\mathbb{P}^1$, and $T^*\mathbb{P}^2$.



- If $\lambda = 0$, all hyperplanes in $\mathbb{C}^d$ passes through the origin, and we see a holomorphic $\mathbb{P}^2 \subset T^*\mathbb{P}^2$:



Section 3

SYZ mirror construction for hypertoric varieties.

Lagrangian torus fibrations on hypertoric varieties

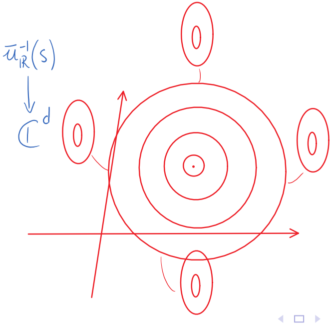
- **Step 1:** Constructing Lagrangian torus fibration on $X_{\theta,\lambda}$.
- Recall we have moment map

$$\bar{\mu}_{\mathbb{R}} : X_{\theta,\lambda} \to \mathbb{R}^d$$

- Symplectic quotient at level $\bar{\mu}_{\mathbb{R}}^{-1}(s) \subset X_{\theta,\lambda}$, $s \in \mathbb{R}^d$:

$$\bar{\mu}_{\mathbb{R}}^{-1}(s)/T^d = \mathbb{C}^d$$

- **Idea:** pulling-back Lagrangian torus fibration from $\mathbb{C}^d$ to $\bar{\mu}_{\mathbb{R}}^{-1}(s)$ and assemble!

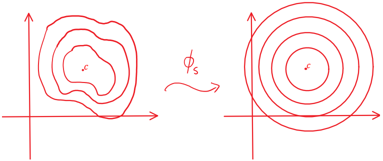


- Such construction was invented by **Harvey-Lawson**, and later generalized by **Gross**, **Goldstein**. It was also used by **Chan-Lau-Leung** to construct SYZ mirrors of toric Calabi-Yau manifolds.
- Problem: $\omega_{Red} \neq \omega_{std}$, standard torus fibration

$$\mathrm{Log} := (\log_t|\zeta_1 - c_1|, \cdots, \log_{t^d}|\zeta_d - c_d|) : \mathbb{C}^d \to (\mathbb{R} \cup -\{\infty\})^d,$$

is not Lagrangian w.r.t. ω_{Red}.
- Solution: Moser's trick:

$$\phi_s^* \omega_{std} = \omega_{Red}$$



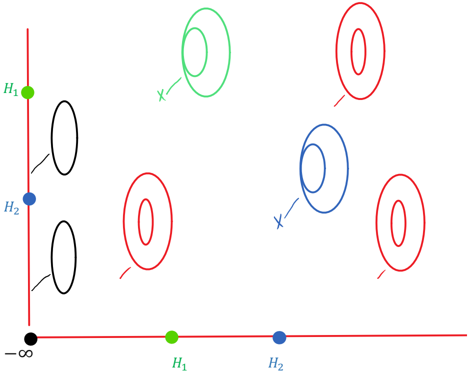
- We get a Lagrangian torus fibration

$$\pi = (\bar{\mu}_{\mathbb{R}}, \mathrm{Log} \circ \bar{\mu}_{\mathbb{C}} \circ \phi) : X_{\theta,\lambda} \to \mathbb{R}^d \oplus (\mathbb{R} \cup \{-\infty\})^d$$

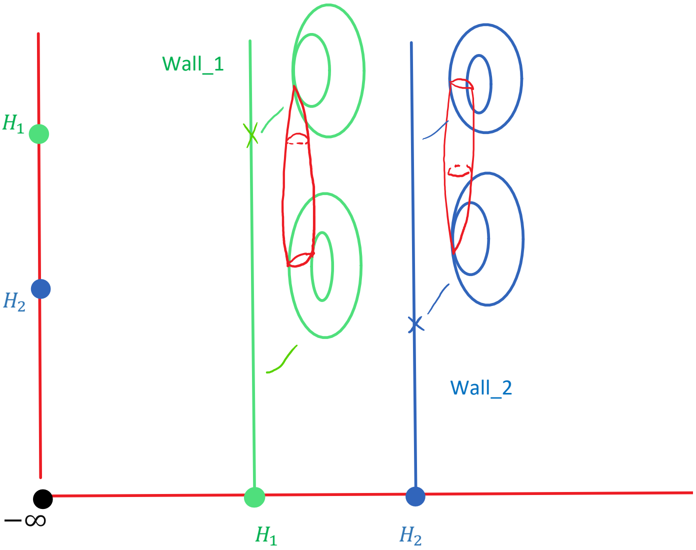
- Remark: taking Log simplifies construction.
- The additional Moser's trick was used by **Auroux-Abouzaid-Katzarkov** to construct SYZ mirrors of blowups of toric varieties along a hypersurface. Our situation differs in that the fibrations constructed have more directions of degeneracy.

Singular loci, walls

- π has singular fibers, mirror construction requires quantum corrections.
- **Step 2:** Analyzing singular loci and walls of π:
- We demonstrate with $\pi : T^*\mathbb{P}^1 \to \mathbb{R} \oplus (\mathbb{R} \cup \{-\infty\})$, where the fibration can be easily visualized.

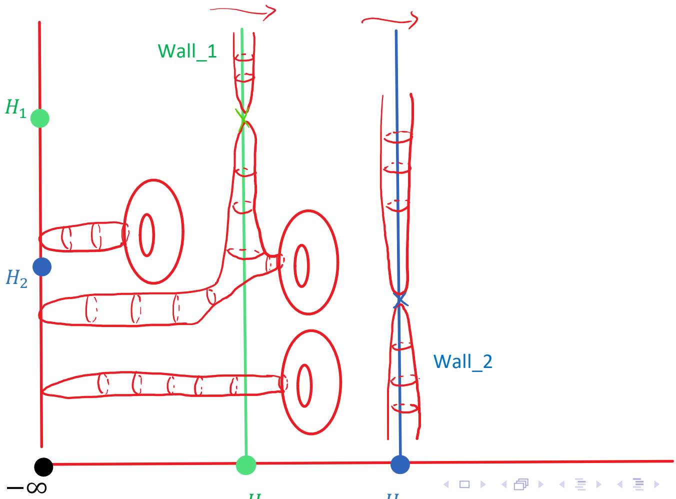


- Walls: fibers over which bound nontrivial holomorphic disc of Maslov index zero.



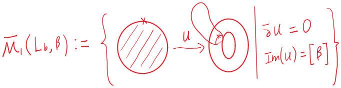
Wall-crossing, quantum corrections

- **Step 3:** Computing the corrected mirror $\check{X}_{\theta,\lambda}$.
- Wall-crossing: When a Lagrangian torus fiber L is isotoped across the walls, discs bounded by L could interact with the discs bounded over the walls.



- Quantum corrections account for the disc interactions and give correction gluing of local charts on the mirror.
- It is defined as the genus 0 open Gromov-Witten invariants n_β, $\beta \in \pi_2(X_{\theta,\lambda}, L_b)$

$$n_\beta := ev_*([\overline{\mathcal{M}}_1(L_b,\beta)]^{vir}) \in H_n(L_b,\mathbb{Q}) = \mathbb{Q}.$$



- Computing n_β for L_b in different chambers gives $X^\vee_{\theta,\lambda}$.
- Example: the mirror of $T^*\mathbb{P}^1$ is the subvariety of $(u_1, v_1, u_2, v_2, \zeta) \in \mathbb{C}^4 \times \mathbb{C}^\times$ defined by

$$\begin{aligned} u_1 v_1 &= 1 + \zeta, \\ u_2 v_2 &= (1+\zeta)(1+\zeta^{-1}), \\ u_1 v_2 &= 1. \end{aligned}$$

partially compactified to account for the Floer theory of the singular fibers.

Multiplicative hypertoric varieties

- The mirrors of hypertoric varieties are in fact multiplicative hypertoric varieties.
- GIT quotient of $(T^*\mathbb{C}^n \setminus \{z_i w_i = 1\}, \frac{dz_i \wedge dw_i}{1 + z_i w_i}) \to (\mathbb{C}^\times)^n$.
- It was discovered by **Mcbreen-Shenfeld** that the quantum connection on equivariant quantum cohomology $H^*_{T^d \times \mathbb{C}^\times}(X_{\theta,\lambda}, \mathbb{C})$ can be identified with certain Gauss-Mannin connection on multiplicative $X_{\theta,\lambda}$.
- They predicated multiplicative $X_{\theta,\lambda}$ to be mirror to $X_{\theta,\lambda}$.
- The mirrors we constructed can be identified with multiplicative hypertoric varieties.