Stability of patterns in reaction-diffusion equations

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Abstract

These are lecture notes associated with the 45-minute talk "Stability of patterns in reaction-diffusion equations," given at the BU/Keio Workshop in Dynamical Systems, during June 25-29, 2018, at Boston University. The abstract of the talk was:

"Reaction-diffusion equations model a wide variety of chemical and biological processes. Such systems are well known for exhibiting patterns, such as traveling waves and spatially- and/or temporally-periodic structures. One important property of such solutions is whether or not they are stable, which is important because it is typically only the stable solutions that are observed in real world settings. In this talk, I will discus the difference between spectral, linear, and nonlinear stability, and highlight some key methods for analyzing stability."

These notes also contain associated exercises.

1 Introduction

This lecture is focused on understanding the stability of patterns in reaction-diffusion equations, which have the form

$$u_t = Du_{xx} + f(u). \tag{1.1}$$

In the above equation, $u = u(x,t) \in \mathbb{R}^n$, $x \in \mathbb{R}$, and $t \geq 0$. The diffusion matrix $D \in \mathbb{R}^{n \times n}$ is assumed to be diagonal with positive entries. Such equations appear in a wide variety of applications, including chemistry and biology.

These equations (and the physical systems they model) exhibit a striking variety of patterns, which are also often referred to as coherent structures or nonlinear waves. Examples include pulses, with profiles that qualitatively resemble $\text{sech}(x)$, fronts, with profiles that qualitatively resemble $\tanh(x)$, spatially periodic solutions including wave trains of the form $e^{i(kx+\omega t)}$, and more complicated structures such as defects [SS04].

Note that such patterns can also travel. For example, a traveling wave is a solution that satisfies $u_{tw}(x,t) = q_{tw}(x - ct)$ for some function q_{tw}, where c is referred to as the wavespeed. Thus, one can change variables in (1.1) to the moving (ξ, t) frame, and then $q_{tw}(\xi)$ will be a stationary solution.

The purpose of this lecture is to understand some issues related to the stability of such solutions. Stability means, roughly speaking, that if the system starts with an initial condition near that particular solution, then the system will stay near it for all time. Let us assume that our solution of interest, $q(x)$, is a stationary solution of (1.1). We can then make the Ansatz

$$u(x,t) = q(x) + v(x,t),$$

where we think of v as representing the perturbation of q. Inserting this into (1.1), we find

$$v_t = Dv_{xx} + df(q(x))v + [f(q(x) + v(x,t)) - f(q(x)) - df(q(x))v] =: \mathcal{L}v + N(v), \tag{1.2}$$

where $\mathcal{L} = D\partial_x^2 + df(q(x))$ is the linear part and N collects the remaining nonlinear terms. We will focus on local stability, which means we will assume that $v(x,0)$ is small. If the resulting solution of (1.2) decays to zero (or at least does not grow), then we say that q is stable. Equivalently, we say that the zero solution of (1.2) is stable. The following makes this more precise. In the below definition, X represents an appropriate Banach space, such as $L^2(\mathbb{R})$.

Definition 1.1. *The solution $v(t) \equiv 0$ of (1.2) is said to be* **stable** *if, given any $\epsilon > 0$, there exists a $\delta = \delta(\epsilon) > 0$ such that, for all initial data $v_0 \in X$ with $\|v_0\|_X \leq \delta$, the corresponding solution to (1.2) satisfies $\|v(t)\|_X \leq \epsilon$ for all $t \geq 0$. If in addition there exists a δ^* such that for all initial conditions with $\|v_0\|_X < \delta^*$ the corresponding solution satisfies* $\lim_{t\to\infty} \|v(t)\|_X = 0$*, then the zero solution is said to be* **asymptotically stable**.

Note that the choice of Banach space is very important. It is possible for solutions to be stable with respect to one Banach space, but unstable with respect to another.

Our main goal will be to discuss some key notions associated with stability – spectral, linear, and nonlinear stability, as well as some of the main techniques used in conducting stability analysis.

1.1 Exercises

1.1.1 Standing wave of a reaction diffusion equation

This example comes from [San], which can be found at `http://www.dam.brown.edu/people/sandsted/documents/evans-function-example.pdf`. Consider the nonlinear equation $u_t = u_{xx} - u + u^3$, $u \in \mathbb{R}$, $x \in \mathbb{R}$, which has an explicit standing pulse given by $a(x) = \sqrt{2}\text{sech}(x)$. Confirm that this is indeed a solution and show that, near this standing pulse, the PDE can be written $v_t = Lv + N(v)$, where $Lv = v_{xx} + (6\text{sech}^2(x) - 1)v$ and $N(v) = 3u_* v^2 + v^3$. Thus, the linearization of the PDE at this pulse is $v_t = v_{xx} + (6\text{sech}^2(x) - 1)v$.

1.1.2 Traveling wave of the bistable equation

The bistable equation is given by $u_t = u_{xx} - u(1-u)(1+u)$, where $\mu \in (0,1)$, and it has an explicit traveling wave given by

$$q(\xi) = \frac{1}{1 + e^{-\frac{1}{\sqrt{2}}\xi}}, \qquad c = \sqrt{2}(\mu - 1/2), \qquad \text{if} \quad \mu \in (0, 1/2].$$

If $\mu \in [1/2, 1)$, then the explicit traveling wave is given by the above formula with (ξ, c) replaced by $(-\xi, -c)$. Confirm that, in terms of the moving coordinate frame $(\xi, t) := (x - ct, t)$, this is indeed a stationary solution. Show that, with respect to these coordinates, the dynamics near the traveling wave can be written $v_t = v_{\xi\xi}+cv_\xi+[2(1+\mu)q-\mu-3q^2]v+[(1+\mu)-3q^2]v^2-v^3$, so $Lv = v_{\xi\xi}+cv_\xi[2(1+\mu)q-\mu-3q^2]v$ and $N(v) = [(1+\mu)-3q^2]v^2-v^3$. More information about the stability of the traveling wave in the bistable equation can be found, for example, in [Xin00].

2 Spectral Stability

Since we are interested in local stability, which means that v is small (at least initially), it is reasonable to expect the linear terms to dominate the nonlinear ones in (1.2), at least for short times. Thus, for the moment we focus on the linear part of the equation.

To begin, consider the finite-dimensional linear equation

$$u_t = Au, \qquad u \in \mathbb{R}^n, \qquad A \in \mathbb{R}^{n\times n}, \qquad t \in \mathbb{R}. \tag{2.1}$$

Recall that the behavior of solutions to this equation is completely determined by the eigenvalues, or spectrum, of the matrix A. In particular, we have the following result.

Proposition 2.1. *The zero solution of (2.1) is stable if and only if $Re\lambda_k \le 0$ for all k and any eigenvalue with zero real part has its geometric multiplicity equal to its algebraic multiplicity. The zero solution is asymptotically stable if and only if $Re\lambda_k < 0$ for all k.*

Note that the condition on the multiplicities of the eigenvalues is needed to prevent algebraic growth in the marginally stable case where there exists an eigenvalue with zero real part. Thus, we would expect to also need $\sup\mathrm{Re}\sigma(\mathcal{L}) \le 0$, where $\sigma(\mathcal{L})$ is the spectrum of the operator $\mathcal{L}$. One key difference between the ODE and PDE cases is that it is significantly harder to compute the spectrum of a (typically) unbounded operator in infinite-dimensions.

Often it is useful to separate the spectrum of $\mathcal{L}$ into the essential spectrum and the point spectrum (eigenvalues). The essential spectrum is relatively easy to compute (see, eg, [Hen81, San02]), but the point spectrum can be quite difficult to locate. Thus, we focus on techniques for determining the latter.

2.1 Techniques for determining spectral stability - computing the point spectrum

In the scalar case ($n = 1$), a powerful tool for computing the spectrum is Sturm-Liouville theory. A nice overview of this can be found in the book [KP13]. It is currently not completely understood if and how this technique can be generalized to higher dimensions, but it is believed that the Maslov Index, a topological invariant similar to a winding number, may be useful. See [Jon88, BJ95, JLM13, BM14].

Perhaps the most widely used tool for computing the spectrum of a linear operator in this context is the Evans function. We will only briefly describe it here, but more details can be found in [AGJ90, San02]. We must find values of λ such that there exists a solution to $\lambda v = \mathcal{L}v$, with $v \in X$. Typically the Banach space X will require that the function v decay to zero as $|x| \to \infty$. If we write the eigenvalue equation as a first order system, we obtain

$$U_x = \mathcal{A}(x,\lambda)U, \qquad U = \begin{pmatrix} v \\ v_x \end{pmatrix}, \qquad \mathcal{A}(x,\lambda) = \begin{pmatrix} 0 & 1 \\ D^{-1}(\lambda - df(q(x))) & 0 \end{pmatrix}.$$

Assuming that the limits $\lim_{x\to\pm\infty} A(x,\lambda) = A_\pm(\lambda)$ are hyperbolic (at least for values of λ to the right of the essential spectrum, where potentially unstable eigenvalues are expected to lie), then any eigenfunction must lie in the intersection of the unstable subspace $E^u_-(x;\lambda)$, coming from $-\infty$, and the stable subspace $E^s_+(x;\lambda)$, coming from $+\infty$. These represent the evolution of the unstable eigenspace of $\mathcal{A}_-(\lambda)$ forwards in x and the evolution of the stable eigenspace of $\mathcal{A}_+(\lambda)$ backwards in x, respectively. If

$$E^u_-(x;\lambda) = \mathrm{span}[u^-_1(x,\lambda),\dots,u^-_k(x,\lambda)], \qquad E^s_-(x;\lambda) = \mathrm{span}[u^+_{k+1}(x,\lambda),\dots,u^+_{2n}(x,\lambda)],$$

then the Evans function is

$$\mathcal{E}(\lambda) = \det[u^-_1(0,\lambda),\dots,u^-_k(0,\lambda),u^+_{k+1}(0,\lambda),\dots,u^+_{2n}(0,\lambda)].$$

Thus, the Evans function maps the complex plane to itself. It can be shown that its zeros, including multiplicity, correspond exactly to eigenvalues of $\mathcal{L}$. The real power of the Evans function comes from the fact that, in many cases, its zeros and other associated properties can be computed relatively explicitly. See [AGJ90, San02] for more details.

3 Linear Stability

We will still focus, for the moment, on the linear equation $v_t = \mathcal{L}v$. If $\mathcal{L}$ were a matrix, then Proposition 2.1 tells us that spectral stability implies linear stability, meaning that if the spectrum of the matrix satisfies the spectral conditions of that Proposition, the solutions to $v_t = \mathcal{L}v$ will actually decay to zero (or remain bounded in the case of eigenvalues with zero real part).

For PDEs where the space X is unbounded, spectral stability does not necessarily imply linear stability. The reason is as follows. At least when $\mathcal{L}$ generates a semigroup $e^{\mathcal{L}t}$ (which is typically the case in our setting), solutions to the linear equation are $v(t) = e^{\mathcal{L}t}v_0$ [EN00]. (If you're not familiar with semigroups, think of them as generalizations of matrix exponentials.) Thus, decay of solutions will be determined by $\|e^{\mathcal{L}t}\|$. In order to relate this to the spectrum of $\mathcal{L}$, we need to use a spectral mapping theorem, which essentially says that

$$\sigma(e^{\mathcal{L}}) \setminus \{0\} = e^{\sigma(\mathcal{L})}.$$

Understanding for which operators $\mathcal{L}$ such a theorem holds is a rather subtle question, and so we won't get into it. See [EN00] for more details.

Rather than relying on the semigroup, another method for determining linear stability is to use a pointwise Green's function. To explain what this is, recall the formula

$$e^{\mathcal{L}t}v_0 = \frac{1}{2\pi \mathrm{i}}\int_\Gamma e^{\lambda t}(\lambda - \mathcal{L})^{-1}v_0 \mathrm{d}\lambda,$$

where Γ is an appropriate contour in the complex plane [EN00]. It is often possible to find an integral kernel, $G(x,y,\lambda)$, that describes the action of the resolvent operator:

$$u(x) = \int_{\mathbb{R}} G(x,y,\lambda)v(y)\mathrm{d}y \qquad \Rightarrow \qquad (\lambda - \mathcal{L})u = v.$$

This allows us to write solutions to $v_t = \mathcal{L}v$ as

$$v(x,t) = \int_{\mathbb{R}} \frac{1}{2\pi \mathrm{i}}\int_\Gamma e^{\lambda t}G(x,y,\lambda)v_0(y)\mathrm{d}\lambda\mathrm{d}y =: \int_{\mathbb{R}} \mathcal{G}(x,y,t)v_0(y)\mathrm{d}y.$$

The function

$$\mathcal{G}(x,y,t) = \frac{1}{2\pi \mathrm{i}} \int_\Gamma e^{\lambda t} G(x,y,\lambda) \mathrm{d}\lambda$$

is called the pointwise Greens function. If, for example, $\mathcal{L} = \partial_x^2$, then $G(x,y,\lambda) = \frac{1}{2\sqrt{\lambda}} e^{-\sqrt{\lambda}|x-y|}$ and

$$\mathcal{G}(x,y,t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{(x-y)^2}{4t}}$$

is just the heat kernel. It is often possible to work with $\mathcal{G}$ directly, to show that $v(x,t)$ must decay to zero as $t \to \infty$. This can be particularly useful when the spectrum is only marginally stable, so at best the solutions to the linear equation will decay algebraically in time. Such techniques are referred to as pointwise Greens function estimates, and they we largely developed in the context of viscous conservation laws by Zumbrun and colleagues [ZH98, Zum11]. See also [BNSZ12, BNSZ14], as well as [BSZ10] for how to extend these results to the time-periodic case.

3.1 An example and some exercises

3.1.1 Example: Standing wave of a reaction diffusion equation

This is a continuation of the example in 1.1.1. We'll work in $X = L^2(\mathbb{R})$. The equation $\lambda u = Lu$ can be written

$$\lambda u = u_{xx} + (6\mathrm{sech}^2(x) - 1)u.$$

Since this is a second-order ODE, we know that for each value of λ there are two independent solutions. One can check that

$$\begin{aligned} u_1(x;\lambda) &= e^{\sqrt{1+\lambda}x}\left[1 + \frac{\lambda}{3} - \sqrt{1+\lambda}\tanh(x) - \mathrm{sech}^2(x)\right] \\ u_2(x;\lambda) &= e^{-\sqrt{1+\lambda}x}\left[1 + \frac{\lambda}{3} + \sqrt{1+\lambda}\tanh(x) - \mathrm{sech}^2(x)\right] \end{aligned}$$

are indeed two independent solutions of the above equation. (They can be found using hypergeometric series.) In order to investigate linear stability, we can determine the resolvent kernel. It turns out that, for this example, we can calculate it explicitly as follows. Suppose we are given a function w and we seek a function u such that $(\lambda - L)u = w$; hence, $u = (\lambda - L)^{-1}w$. We'll use the method of variation of parameters, which means that we assume the function u has the form

$$u(x;\lambda) = v_1(x;\lambda)u_1(x;\lambda) + v_2(x;\lambda)u_2(x;\lambda),$$

and solve for the functions $v_{1,2}$ in terms of w. To do this, we impose the condition that $v_1'u_1 + v_2'u_2 = 0$, which is one equation, and insert the above form of u into the equation $(\lambda - L)u = w$ to obtain a second equation. These two equations can be written

$$\begin{pmatrix} v_1' \\ v_2' \end{pmatrix} = \frac{1}{u_1u_2' - u_2u_1'} \begin{pmatrix} u_2' & -u_2 \\ -u_1' & u_1 \end{pmatrix} \begin{pmatrix} 0 \\ -w \end{pmatrix} = -\frac{9}{2\lambda\sqrt{1+\lambda}(3-\lambda)} \begin{pmatrix} u_2' & -u_2 \\ -u_1' & u_1 \end{pmatrix} \begin{pmatrix} 0 \\ -w \end{pmatrix},$$

where to obtain the final equality we have used the above expressions for $u_{1,2}$ to explicitly calculate the Wronskian $u_1u_2' - u_2u_1'$. We can see immediately that there will be problems if $\lambda \in \{0,3\} \cup (-\infty,-1]$.

Continuing our calculation of $v_{1,2}$ (for λ not in this bad set), we find

$$v_1' = -\frac{9}{2\lambda\sqrt{1+\lambda}(3-\lambda)}u_2 w, \qquad v_2' = \frac{9}{2\lambda\sqrt{1+\lambda}(3-\lambda)}u_1 w.$$

To integrate these expressions, we note that u_1 is well-behaved at $-\infty$ while u_2 is well-behaved at $+\infty$. Hence, we define

$$v_1(x;\lambda) = \frac{9}{2\lambda\sqrt{1+\lambda}(3-\lambda)}\int_x^\infty u_2(y;\lambda)w(y)\mathrm{d}y, \qquad v_2(x;\lambda) = \frac{9}{2\lambda\sqrt{1+\lambda}(3-\lambda)}\int_{-\infty}^x u_1(y;\lambda)w(y)\mathrm{d}y.$$

Inserting these formulas back into the expression for u, one finds that the solution can be written

$$\begin{aligned} u(x) &= \frac{9}{2\lambda\sqrt{1+\lambda}(3-\lambda)}\int_{\mathbb{R}}[u_1(x;\lambda)u_2(y;\lambda)H(y-x) + u_2(x;\lambda)u_1(y;\lambda)H(x-y)]w(y)\mathrm{d}y \\ &=: \int_{\mathbb{R}} G(x,y;\lambda)w(y)\mathrm{d}y. \end{aligned}$$

Therefore, the action of the resolvent operator can be expressed through the integral kernel $G(x,y;\lambda)$. One can now investigate for which values of λ this operator is well-defined and bounded on all of L^2 and prove that $\sigma_{\mathrm{pt}}(L) = \{0,3\}$ and $\sigma_{\mathrm{ess}}(L) = (-\infty,-1]$. If $\lambda = 0$ or $\lambda = 3$, then the kernel of $(\lambda - L)$ is one-dimensional: $-u_1(x;\lambda) = u_2(x;0) = \mathrm{sech}(x)\mathrm{tanh}(x)$ and $u_1(x;3) = u_2(x;3) = \mathrm{sech}^2(x)$, which are the corresponding eigenfunctions. (In each case, the second, linearly independent solution to $\lambda u = u_{xx}$ is a function that's not in L^2.)

Note that one can also compute the Evans function in this example and see that it is

$$\mathcal{E}(\lambda) = -\frac{2}{9}\lambda\sqrt{1+\lambda}(\lambda-3).$$

3.1.2 Exercise: The Laplacian

Use the method of variation of parameters, illustrated in the previous examples, to show that the resolvent kernel for the Laplacian is indeed $G(x,y,\lambda) = \frac{1}{2\sqrt{\lambda}}e^{-\sqrt{\lambda}|x-y|}$.

3.1.3 Exercise: pulses of scalar reaction-diffusion equations are always linearly unstable

Consider a scalar reaction diffusion equation

$$u_t = u_{xx} + f(u),$$

and suppose that there exist a pulse-type solution, ie a function $u_*(x)$ such that $u_*'(x) \geq 0$ for all $x < x_0$, $u_*'(x) \leq 0$ for all $x > x_0$, and $u_*(x) \to 0$ exponentially fast as $|x| \to 0$. (As an example, consider §3.1.1, above.) Show that this solution is linearly unstable using the following steps.

- Show that the linearization is $v_t = v_{xx} + Df(u_*(x))v$.
- Show that $u_*'(x)$ is an eigenfunction with eigenvalue zero.
- Apply Sturm-Liouville theory to the equation $\lambda u = Lu$.
- Explain why this implies that the pulse is linearly unstable.

4 Nonlinear Stability

Now we seek to understand when solutions to the full nonlinear equation (1.2) decay to zero. Assuming one has already shown linear stability, one can often extend that decay to the nonlinear equation via Duhamel's formula, which is sometimes also called variation of constants or parameters. In particular, using the semigroup $e^{\mathcal{L}t}$, solutions to (1.2) can be written

$$v(t) = e^{\mathcal{L}t}v_0 + \int_0^t e^{\mathcal{L}(t-s)}N(v(s))\mathrm{d}s.$$

Alternatively, using the pointwise Greens function, we have

$$v(x,t) = \int_{\mathbb{R}} \mathcal{G}(x,y,t)v_0(y)\mathrm{d}y + \int_0^t \int_{\mathbb{R}} \mathcal{G}(x,y,t-s)N(v(y,s))\mathrm{d}y\mathrm{d}s.$$

It is ofter nontrival to prove nonlinear stability, even if one has detailed information about linear stability. See [Zum11, BNSZ12, BNSZ14, BSZ10] for some examples. Also, it is possible for a solution to be linearly unstable, but nonlinearly stable (and vice versa) [BGS09].

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