Rotating Wave Solutions to Lattice Dynamical Systems

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Rotating Waves

 

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Spiral Waves

Reaction-Diffusion Equations and Euclidean Symmetry

- Investigations of spiral waves have primarily focused on reaction-diffusion equations (RDEs) such as:

$$\frac{\partial u}{\partial t} = D \cdot \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) + \mathcal{F}(u), \tag{1}$$

where $D > 0$, $u = u(x, y, t) : \mathbb{R}^2 \times \mathbb{R}^+ \to \mathbb{R}^n$ and $\mathcal{F} : \mathbb{R}^n \to \mathbb{R}^n$

- System (1) has the property that if $u(x, y, t)$ is a solution then so is

$$\tilde{u}(x, y, t) = u(x \cos\theta - y \sin\theta + p_1, x \sin\theta + y \cos\theta + p_2, t)$$

- Many investigations lately concerned with breaking this symmetry property

The Retracting Tip Phenomenon

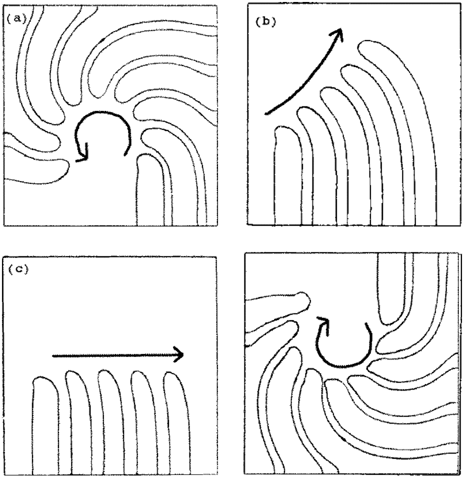

Image taken from Ashwin, Melbourne, and Nicole (1999)

Lattice Dynamical Systems

- For a spatial step size $h > 0$, one uses the approximation

$$\frac{\partial^2 u}{\partial x^2}(x, y) \approx \frac{u(x+h, y) + u(x-h, y) - 2u(x, y)}{h^2},$$

  and an analogous approximation for $\partial^2 u/\partial y^2$
- Moving to the spatial grid $x = ih$ and $y = jh$ for $i, j \in \mathbb{Z}$, (1) gives the discrete spatial approximation

$$\frac{d}{dt}u(ih, jh, t) \approx \alpha \sum_{i',j'} (u(i'h, j'h, t) - u(ih, jh, t)) + \mathcal{F}(u(ih, jh, t))$$

  for each $i, j \in \mathbb{Z}$ and $\alpha = \frac{D}{h^2}$
- Throughout we will write $u(ih, jh, t) = u_{i,j}(t)$ to emphasize that this is now an ODE

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- Discrete Space (1D): Taking $\mathcal{F}(x) = x(1-x)(x-a)$ and $0 < \alpha \ll 1$ there is an interval about $a = 1/2$ which gives propagation failure
- Discrete Space (2D): Cahn, Mallet-Paret and Van Vleck (1998) have shown that propagation success/failure depends on the direction of propagation

Lambda-Omega Reaction-Diffusion Equations

- Howard and Kopell (1979) introduced so-called Lambda-Omega RDEs in terms of a single complex variable $z(x, y, t) : \mathbb{R}^2 \times \mathbb{R}^+ \to \mathbb{C}$ of the form:

$$\frac{\partial z}{\partial t} = D\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right) + z[\lambda(|z|) + \mathrm{i}\omega(|z|)]$$

- Specific forms of λ and ω functions are taken to induce oscillatory behaviour when $D = 0$
- Well-known to arise as the lowest order perturbation of any reaction-diffusion system near a Hopf bifurcation (Cohen, Neu, and Rosales, 1978)
- Typical examples are

$$\lambda(r) = \pm 1 \mp r^2, \qquad \omega(r) = constant$$

Lambda-Omega Lattice Differential Equations

- My work focusses on the analogous Lambda-Omega LDS, given as

$$\dot{z}_{i,j} = \alpha \sum_{i',j'} (z_{i'j'} - z_{i,j}) + z_{i,j}[\lambda(|z_{i,j}|) + \mathrm{i}\omega(|z_{i,j}|, \alpha)]$$

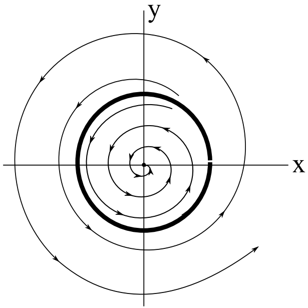
Hypothesis

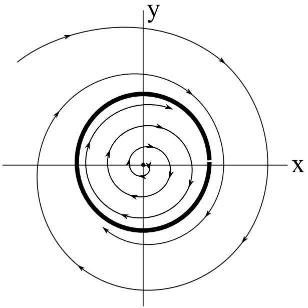
- *(1)* $\lambda : [0, \infty) \to \mathbb{R}$ *is continuously differentiable and there exists some* $a > 0$*, with the property that* $\lambda(a) = 0$ *and* $\lambda'(a) \neq 0$*.*
- *(2)* $\omega = \omega(R, \alpha) : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ *is continuously differentiable in both its arguments such that*

$$\omega(R, \alpha) - \omega(a, \alpha) = \alpha\omega_1(R, \alpha), \tag{2}$$

for some function $\omega_1(R, \alpha)$ *which is continuously differentiable on the same domain with* $\omega_1(a, \alpha) = 0$ *for all* $\alpha \in \mathbb{R}$*.*

Two Cases for the Uncoupled System





Typical phase portraits of the uncoupled system ($\alpha = 0$) and some nearby trajectories. There are two cases: (Left) locally repelling when $\lambda'(a) > 0$ and (Right) locally attracting when $\lambda'(a) < 0$.

Reduction to Polar Coordinates

- Writing $z_{i,j} = r_{i,j} e^{\mathrm{i}(\omega(a,\alpha)t + \theta_{i,j})}$, the Lambda-Omega LDS becomes

$$\dot{r}_{i,j} = \alpha \sum_{i',j'} [r_{i',j'} \cos(\theta_{i',j'} - \theta_{i,j}) - r_{i,j}] + r_{i,j}\lambda(r_{i,j}),$$
$$\dot{\theta}_{i,j} = \alpha \sum_{i',j'} \frac{r_{i',j'}}{r_{i,j}} \sin(\theta_{i',j'} - \theta_{i,j}) + \alpha\omega_1(r_{i,j}, \alpha)$$

- Rotating waves will satisfy the symmetry condition:

$$z_{j,i-1}(t) = e^{\mathrm{i}\frac{\pi}{2}} \cdot z_{i,j}(t)$$

- Interested in rotating waves for $\alpha \to 0^+$

The Phase System

- When $\alpha = 0$ the radial components completely decouple leaving one to solve

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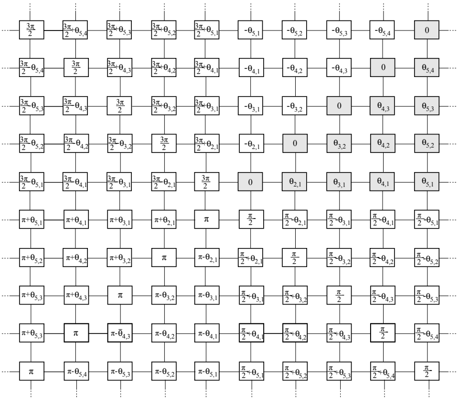
- This phase system can be shown to possess a rotating wave solution

The Phase Solution



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- Restrict to $1 \leq j < i$

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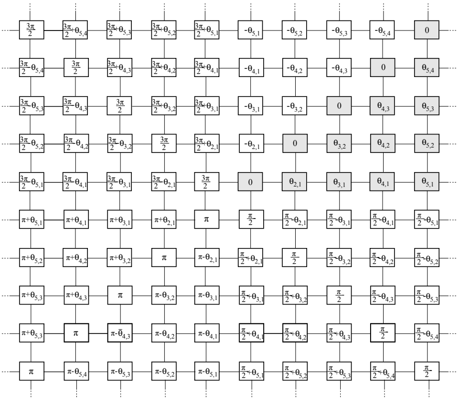

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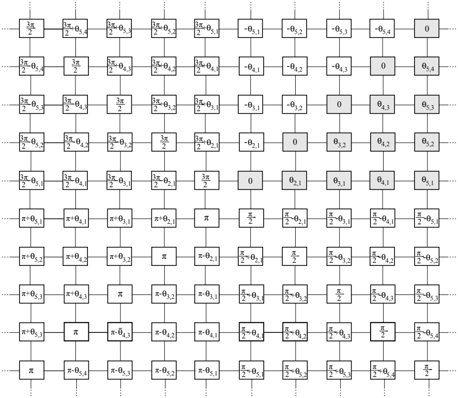

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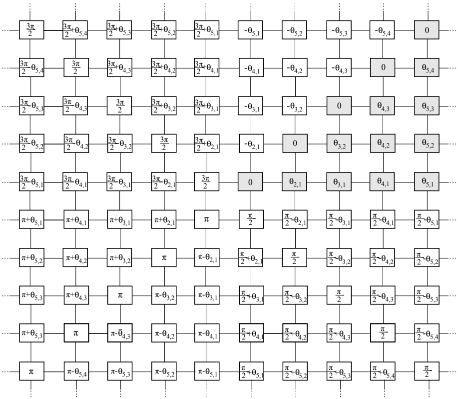

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- Use symmetry extensions to extend over entire lattice

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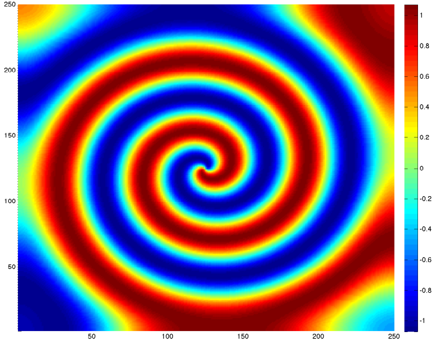
- Set up a mapping whose roots lie in one-to-one correspondence with the steady-states of the polar decomposition
- Can prove that there exists a spiral wave solution to the Lambda-Omega system for sufficiently small $\alpha > 0$

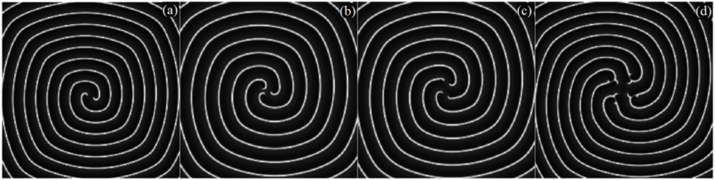
A Spiral Wave Solution



Contour plot of real part of the solution on a 250×250 lattice with $\alpha = 1$, $\lambda(R) = 1 - R^2$ and $\omega(R, \alpha) = 1 + 0.5\alpha R^2$

Necessity that ω is 'Almost Constant'

Discussion



- From the work of Ermentrout and Paullet (1998) on the finite square lattice, one expects the solution to exist for all $\alpha > 0$ when $\omega(R, \alpha)$ is independent of R
- Extension to existence of multi-armed spirals is significantly different and potentially more difficult
- Still wish to examine how the dynamics of discrete space solutions compares to continuous space solutions

Stability

- Even in the continuum setting very little is known about the stability of spiral waves
- In the small $\alpha > 0$ parameter region the system becomes an infinite dimensional fast-slow dynamical system:

$$\dot{r}_{i,j} = \alpha \sum_{i',j'} [r_{i',j'} \cos(\theta_{i',j'} - \theta_{i,j}) - r_{i,j}] + r_{i,j}\lambda(r_{i,j}),$$
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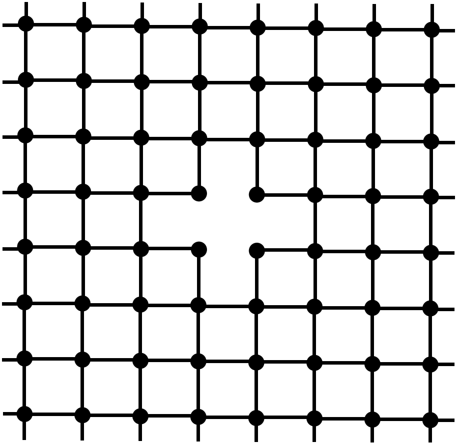
- Current work is attempting to prove stability by determining the existence of an exponentially stable integral manifold for the radial components, and then work only with the phase components to obtain algebraic decay back to equilibrium

Related Work

- The leading order dynamics on the integral manifold are governed by the flow
$$\dot{\theta}_{i,j} = \sum_{i',j'} \sin(\theta_{i',j'} - \theta_{i,j}), \tag{3}$$
which is an infinite-dimensional Kuramoto-style system of coupled oscillators
- Linearizing (3) about a steady-state $\{\bar{\theta}_{i,j}\}_{(i,j)\in\mathbb{Z}^2}$ results in the linear operator acting on the sequences $x = \{x_{i,j}\}$ by
$$[Lx]_{i,j} = \sum_{i',j'} \cos(\bar{\theta}_{i',j'} - \bar{\theta}_{i,j})(x_{i',j'} - x_{i,j})$$
- Natural underlying graph theoretic meaning

Graph Structure of the Rotating Wave



Thank you all for listening!

Questions?

1. J. Bramburger. Rotating wave solutions to lattice dynamical systems I: The anti-continuum limit, *J. Dyn. Differ. Equ.*, at press.
2. J. Bramburger. Rotating wave solutions to lattice dynamical systems II: Persistence results, *J. Dyn. Differ. Equ.*, at press.
3. J. Bramburger. Stability of infinite systems of coupled oscillators via random walks on weighted graphs, *T. Am. Math. Soc.*, at press.