A Sierpinski Mandelbrot Spiral

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Boston University

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Outline

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What Do I Do?

- I study topological structures of bifurcation diagrams for singularly perturbed complex rational maps.

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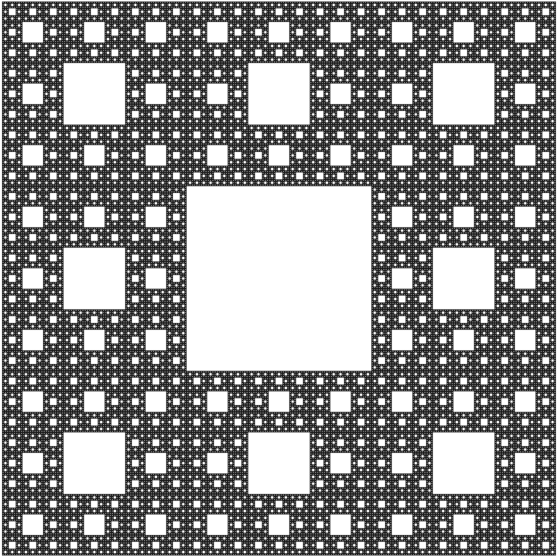
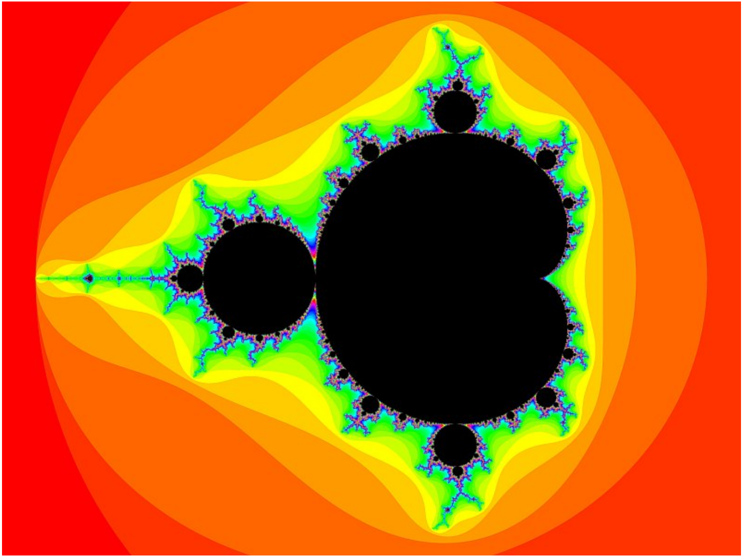
- I study topological structures of bifurcation diagrams for singularly perturbed complex rational maps.
- I am mainly interested in Sierpinski holes and Mandelbrot sets.

Sierpinski Carpet Fractal



Mandelbrot Set



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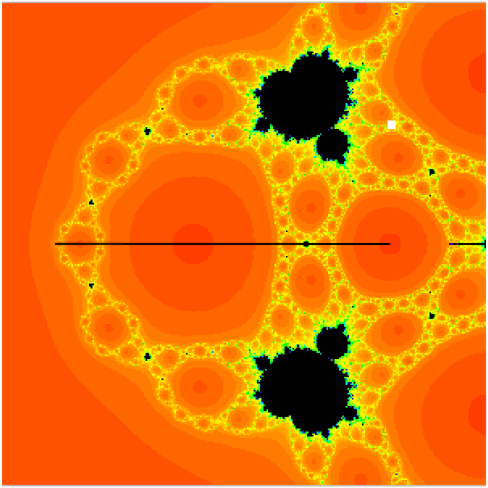

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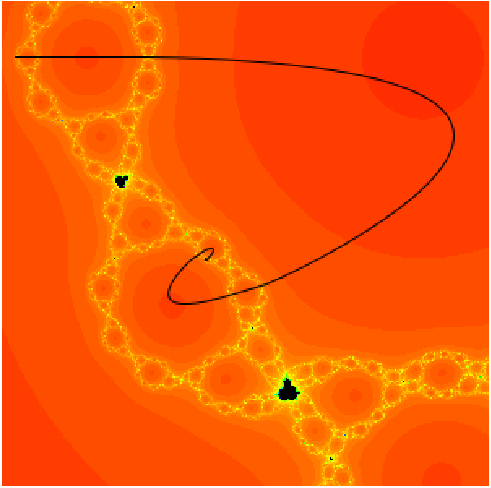
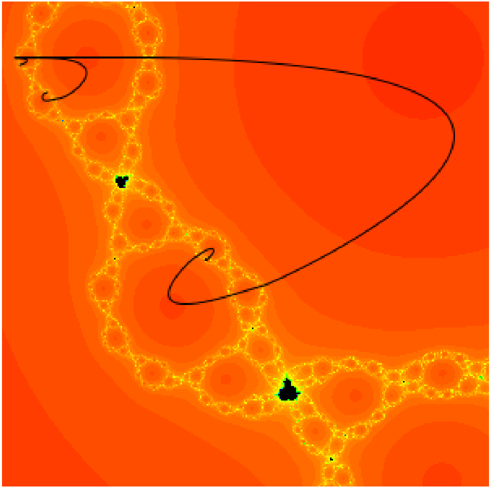


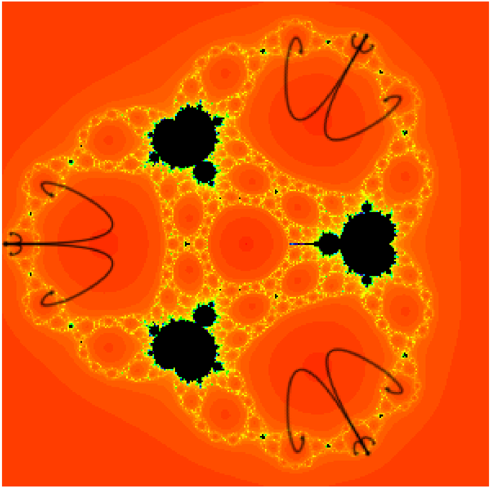
Arcs and Spirals

We will find infinitely many Sierpinski Mandelbrot spirals.



Arcs and Spirals

We will find a Sierpinski Mandelbrot hydra.



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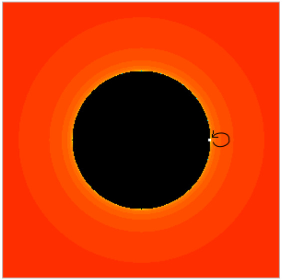

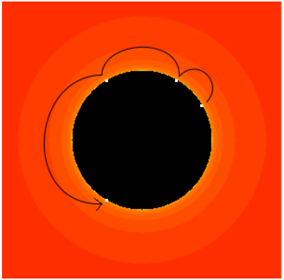

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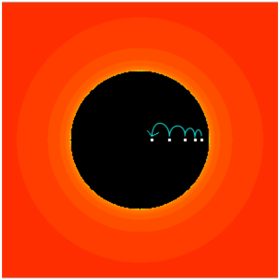

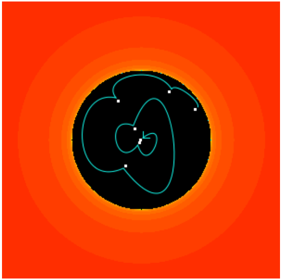

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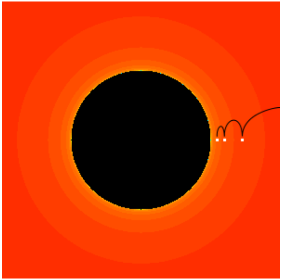

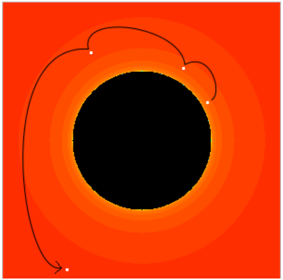

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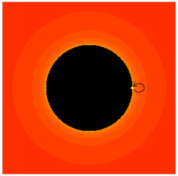
Drawing the Dynamical Plane



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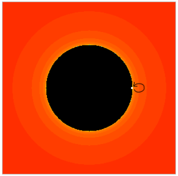

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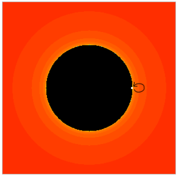

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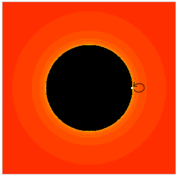

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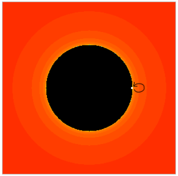

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- the Fatou Set, or $\mathcal{F}(F)$. This is the complement of $\mathcal{J}(F)$.

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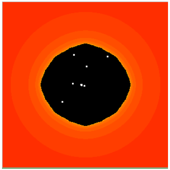

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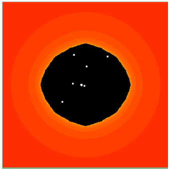

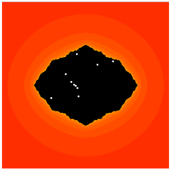

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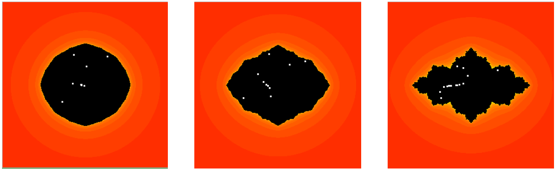

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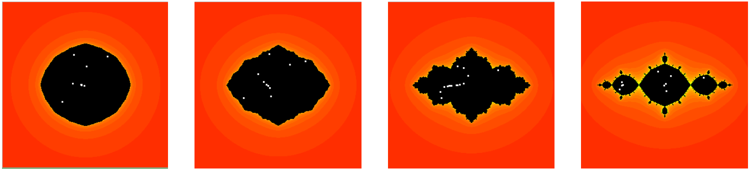

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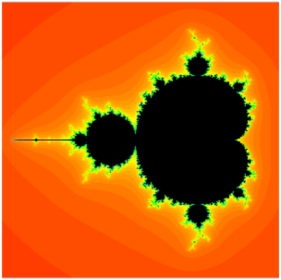
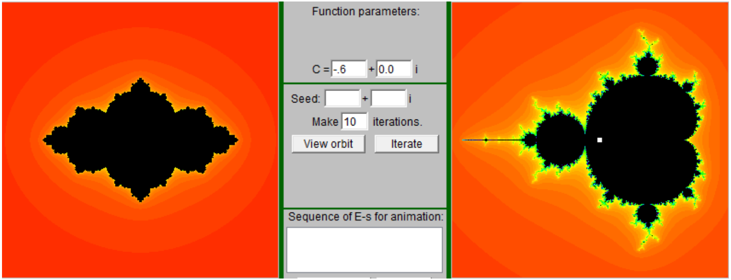


$\mathcal{J}(F_\lambda)$ Depends on λ



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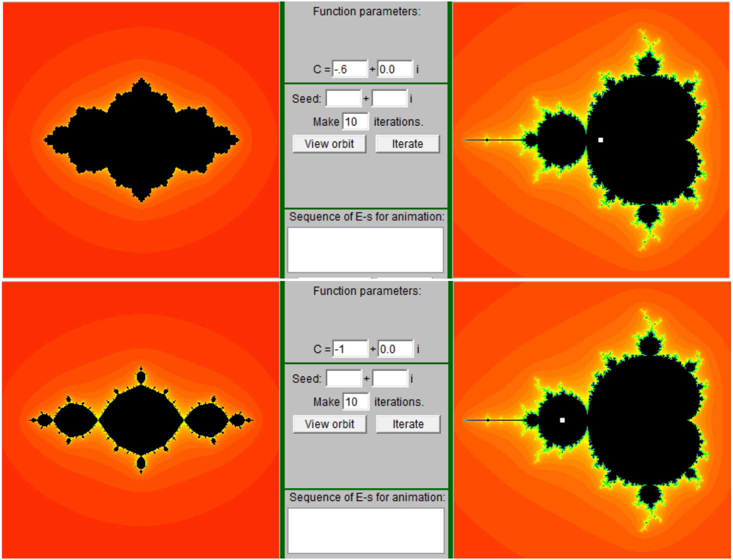

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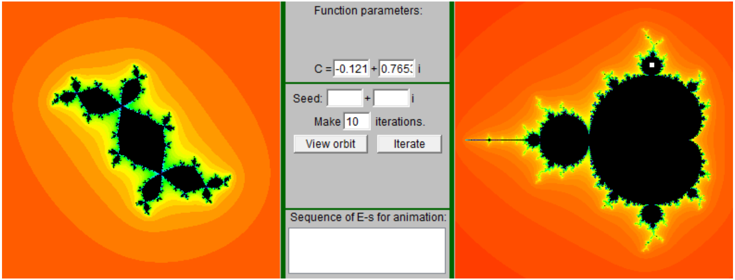

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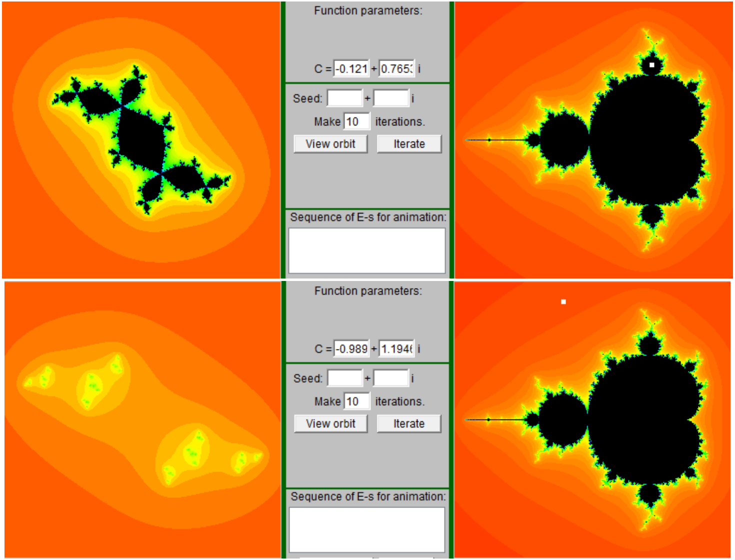

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- Critical values are the next iterates of critical points.

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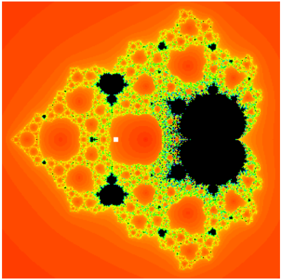
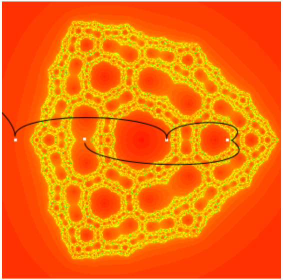


The Basin and Trap Door



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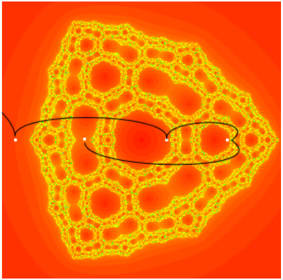

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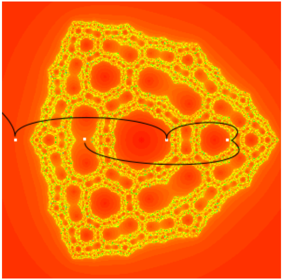

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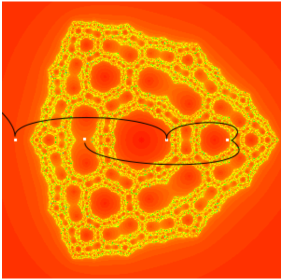

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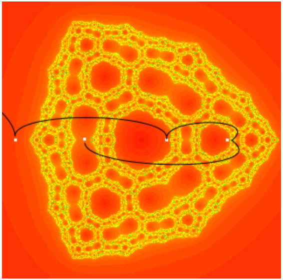

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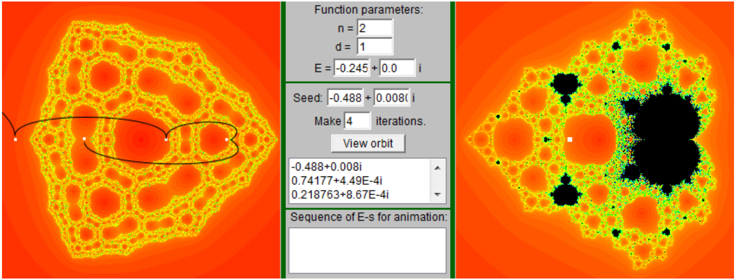
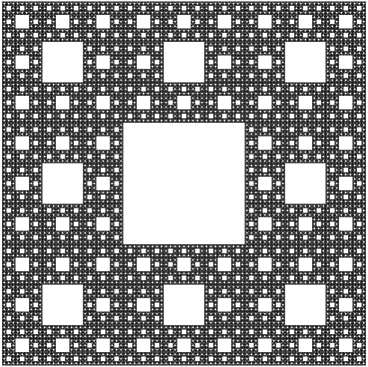
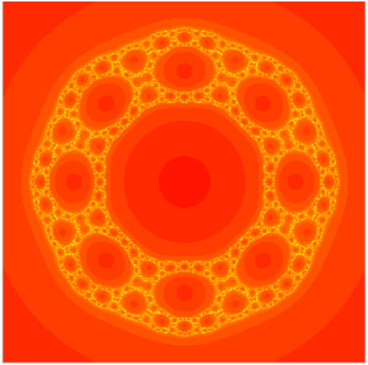
Sierpinski Holes



- $\mathcal{J}(F)$ is homeomorphic to a Sierpinski carpet fractal for λ in a Sierpinski hole.

Sierpinski Carpet Fractal





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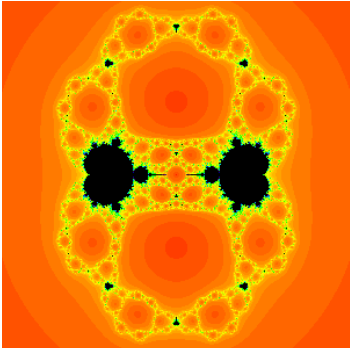
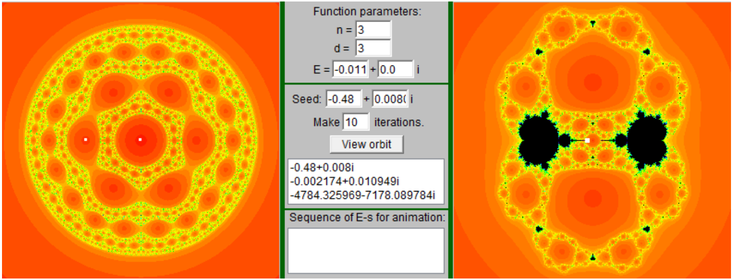


The McMullen Domain



- $\mathcal{J}(F)$ is a Cantor set of simple closed curves.

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In the parameter plane, we can have:

- black for the Mandelbrot set,
- reddish-orange for the McMullen domain, Sierpinski holes, and Cantor set locus, and
- yellow is actually either black or not-black - zoom in.

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A Typical Dynamical Plane

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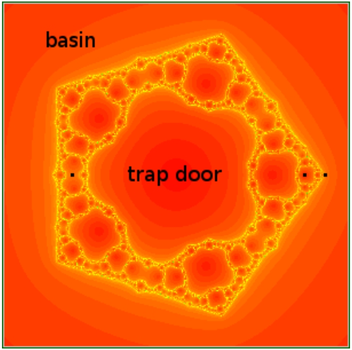

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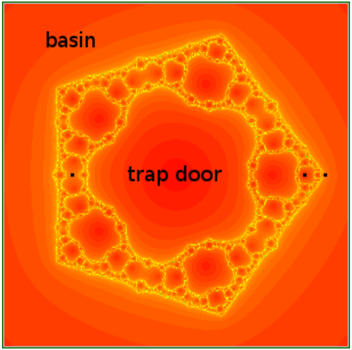
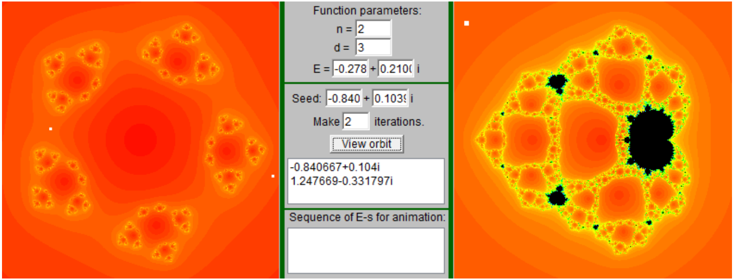


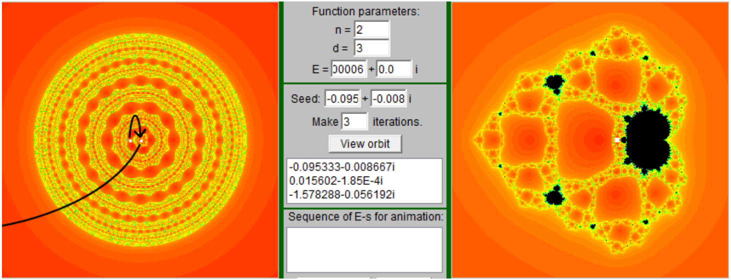
- There exist 5 critical points, critical values, and prepoles.
- We can classify all of the regions in the parameter plane.

The Cantor Set Locus



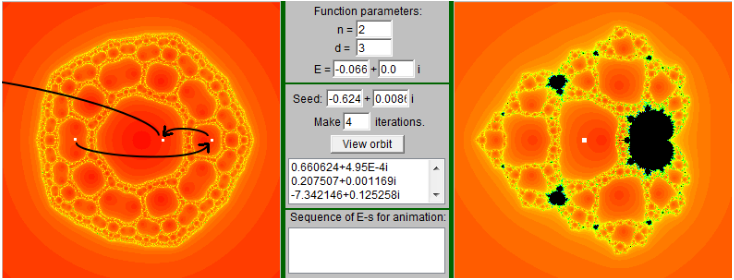
- c^λ lies in B_λ.

The McMullen Domain



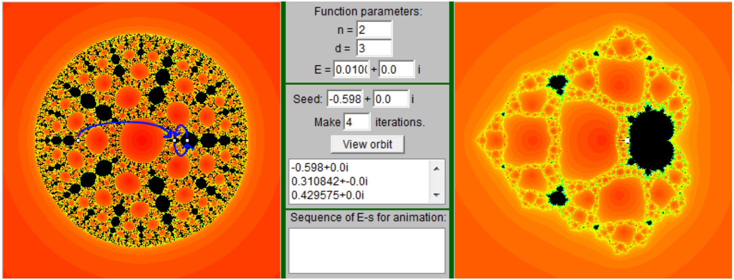
- c^{λ} enters T_{λ} after 1 iteration.

A Sierpinski Hole



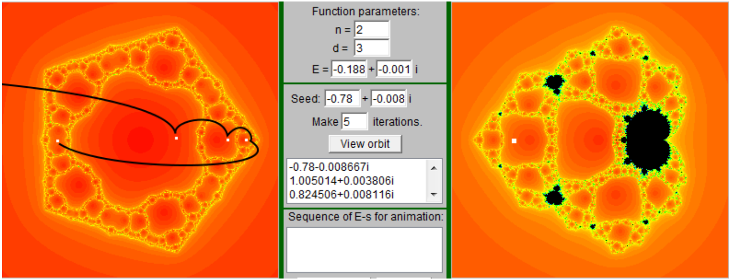
- c^λ enters T_λ after 2 iterations.

A Mandelbrot Set



- c^λ does not escape and is instead trapped in some periodic orbit.

A Sierpinski Hole with Higher Escape Time



- For λ in the next Sierpinski hole to the left:

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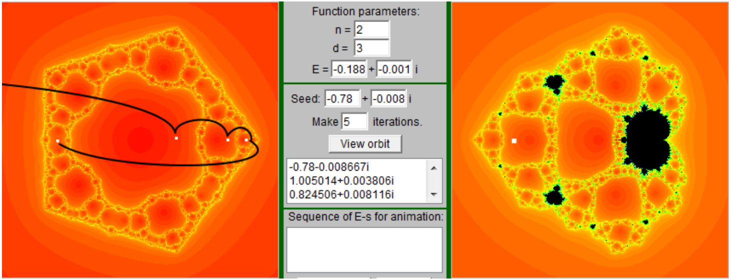
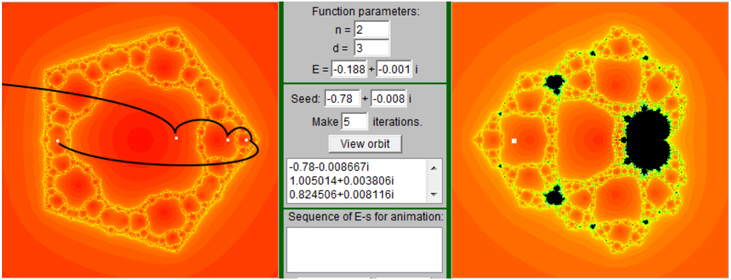


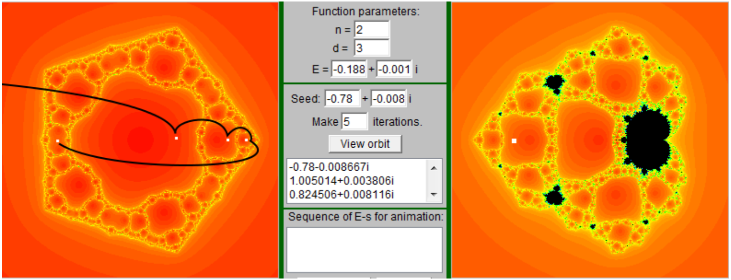
- For λ in the next Sierpinski hole to the left: c^{λ} enters T_{λ} at iteration 3.

A Sierpinski Hole with Higher Escape Time



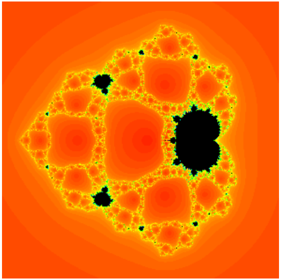
- For λ in the next Sierpinski hole to the left:
  c^{λ} enters T_{λ} at iteration 3.
- What about escape time of the next Sierpinski hole?

A Sierpinski Hole with Higher Escape Time



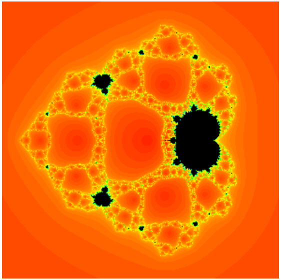
- For λ in the next Sierpinski hole to the left: c^λ enters T_λ at iteration 3.
- What about escape time of the next Sierpinski hole? Anything besides Sierpinski holes?

Many Mandelbrot Sets



- There is the clearly visible principal Mandelbrot set.

Many Mandelbrot Sets



- There is the clearly visible principal Mandelbrot set.
- Also two baby Mandelbrot sets.

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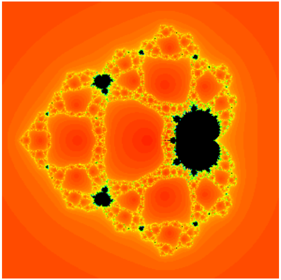
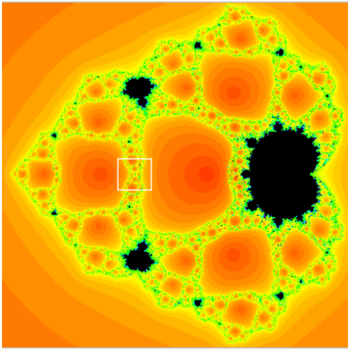


- There is the clearly visible principal Mandelbrot set.
- Also two baby Mandelbrot sets.
- Six more baby Mandelbrot sets.

Zooming In



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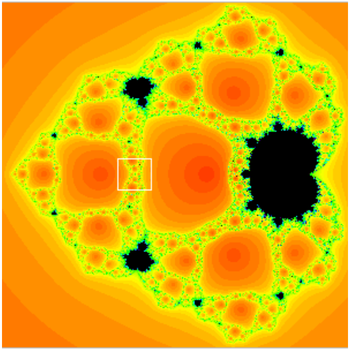
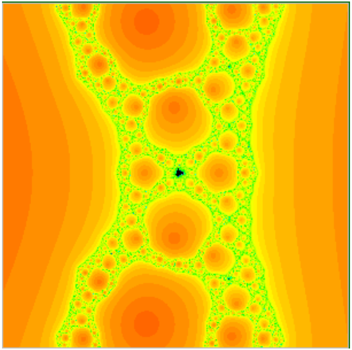
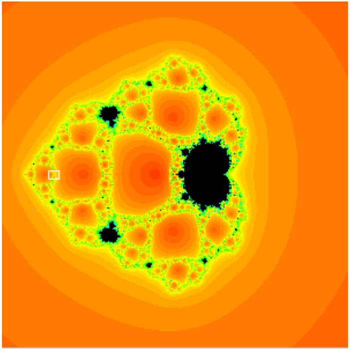




- There is a Mandelbrot between the Sierpinski holes of c^λ escape time 2 and 3.

Further Along $\mathbb{R}^-$



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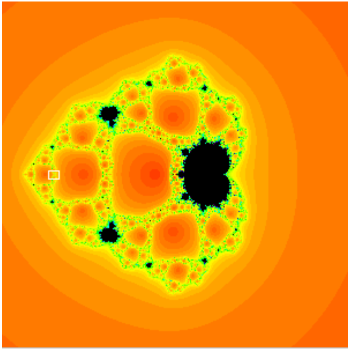

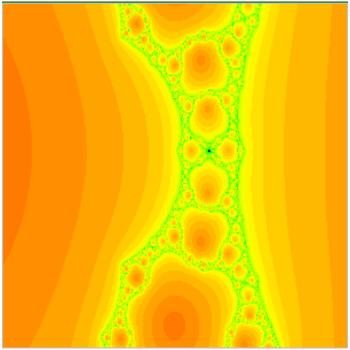

- Looks like another Mandelbrot set between the next pair of Sierpinski holes.

Claim

- There are infinitely many Sierpinski holes along the negative real axis of the parameter plane.

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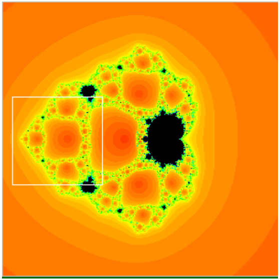

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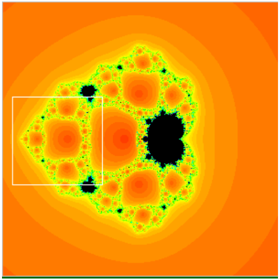

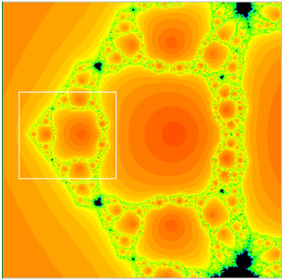

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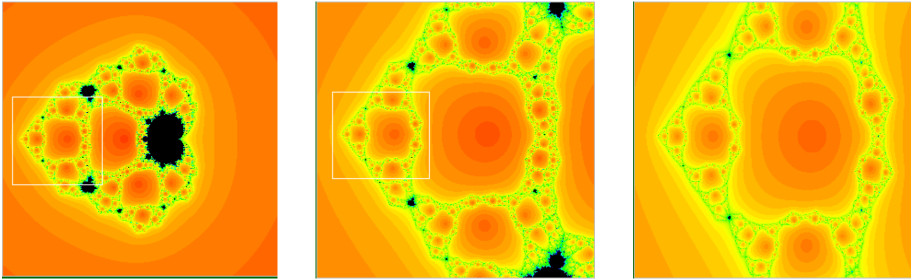

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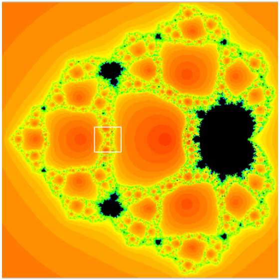

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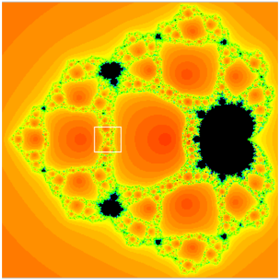

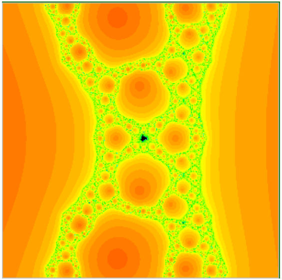

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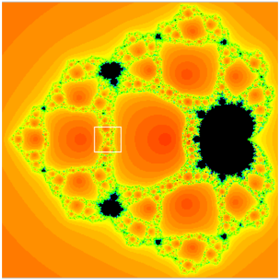

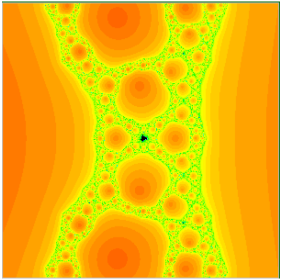

- This set of infinitely many alternating Sierpinski holes and Mandelbrot sets along the negative real axis in the parameter plane is a *Sierpinski Mandelbrot arc*.

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- We restrict λ to some subset of the entire parameter plane, and prove that the dynamical properties hold even if we move λ around.
- Then these dynamical constructs prove the existence of structures in that subset of the parameter plane.

Dynamical Constructs $\implies$ Parameter Structures

- The dynamical constructs are two wedges joined by a circle. The parameter structures are Sierpinski holes and Mandelbrot sets.

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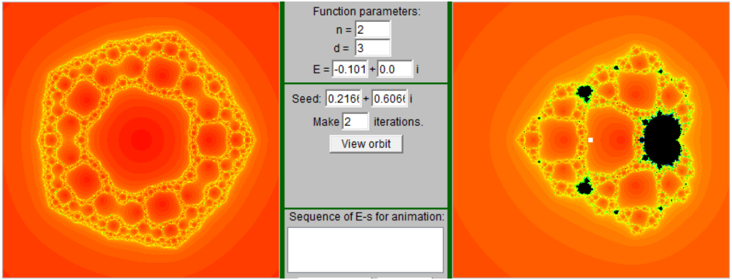
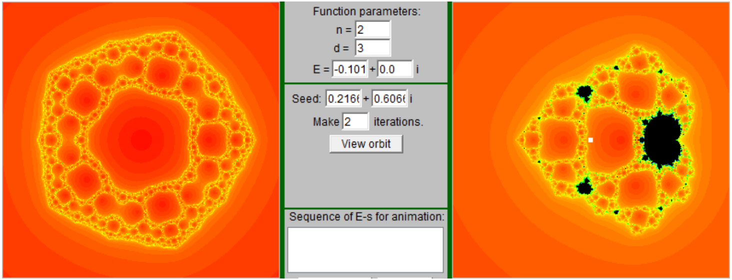
- We need to find a set in the dynamical plane that maps 2-1 over another set. That (plus some other conditions) $\Longrightarrow$ F_λ is a polynomial-like map of degree 2 on those sets. As shown by Douady and Hubbard, that proves the existence of a homeomorphic copy of the Mandelbrot set.

Dynamical Sets



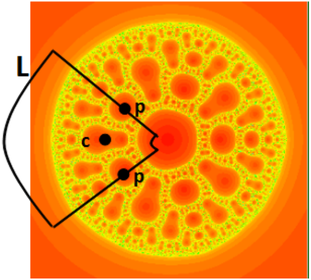
- The left image is the dynamical plane for $n = 2, d = 3$ and λ in a Sierpinski hole on the negative real axis.

Dynamical Sets



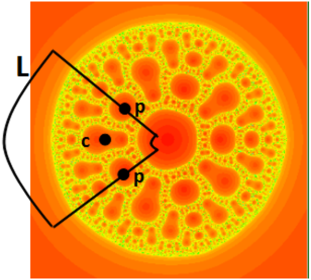
- The left image is the dynamical plane for $n = 2, d = 3$ and λ in a Sierpinski hole on the negative real axis.
- This λ is in the subset of the parameter plane (details).

The Left Wedge L^λ



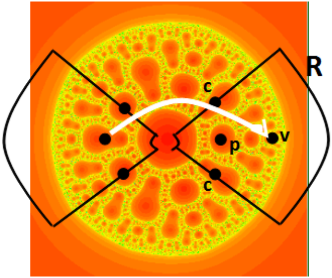
- Let the left wedge, or L^λ, be the closed set as shown.

The Left Wedge L^λ



- Let the left wedge, or L^λ, be the closed set as shown.
- There is one critical point c_0^λ in the interior of L^λ.

The Right Wedge R^λ



- Let R^λ be the symmetric right wedge.

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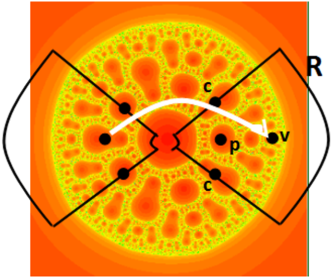

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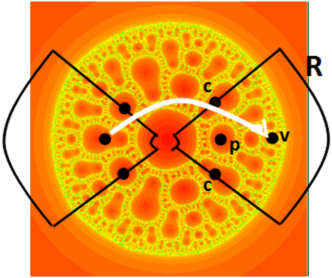
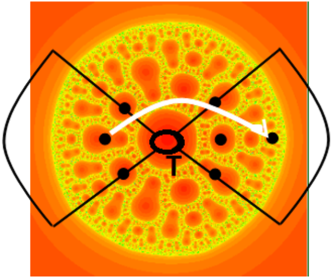


- Let R^λ be the symmetric right wedge.
- There is one prepole p_2^λ in the interior of R^λ.
- The critical point in L^λ maps to the critical value in R^λ.

The (Subset of the) Trapdoor $T_{\mathcal{A}}$



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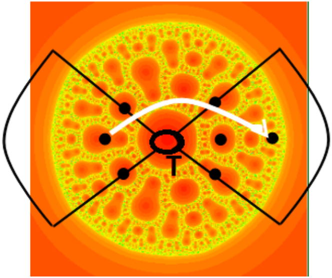

- Let $T_{\mathcal{A}}$ be the closed subset of the trapdoor containing 0 such that $L^{\lambda} \cup T_{\mathcal{A}} \cup R^{\lambda}$ are connected, and they only intersect along boundaries.
- This union of the wedges and $T_{\mathcal{A}}$ will be referred to as the bowtie.

Proposition

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For each λ in that annular region:

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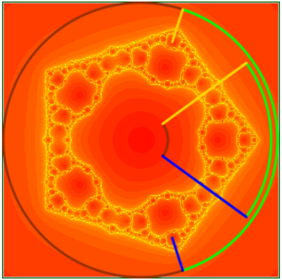
2. F_λ maps L^λ 2-1 over a region that contains the interior of R^λ
3. As λ winds once around the boundary of the the annular region, the critical value $F_\lambda(c_0^\lambda)$ winds once around the boundary of R^λ.

R^λ Contains the Bowtie



- For λ on $\mathbb{R}^-$, the image of R^λ is disjoint from the bowtie.

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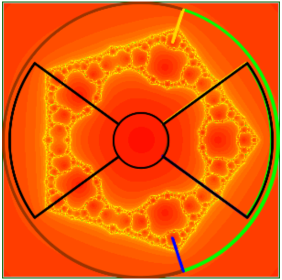
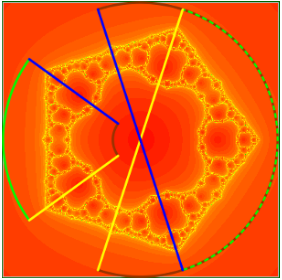


- For λ on $\mathbb{R}^-$, the image of R^λ is disjoint from the bowtie.
- This remains true as we rotate λ to the edges of the λ region.

L^λ Contains R^λ



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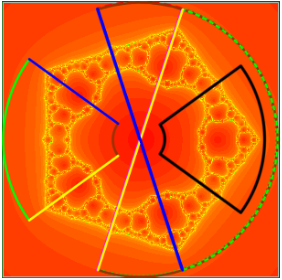

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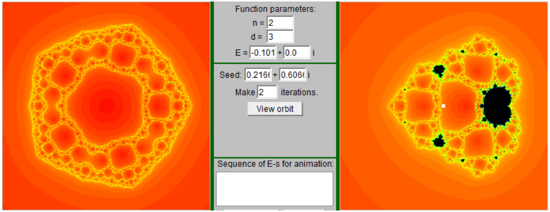
Drawing a Picture



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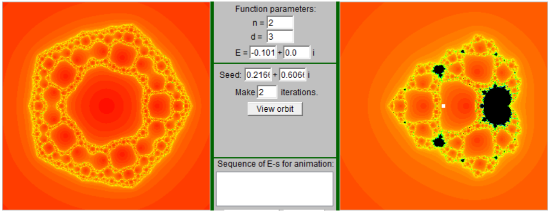

- We can "put a bowtie" on the dynamical plane.

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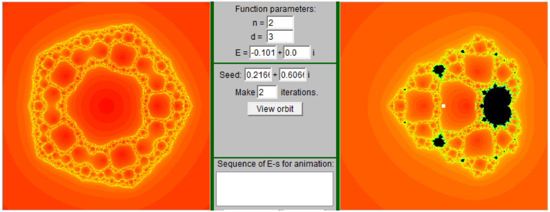

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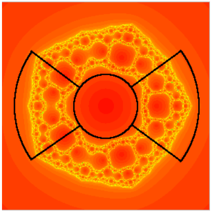

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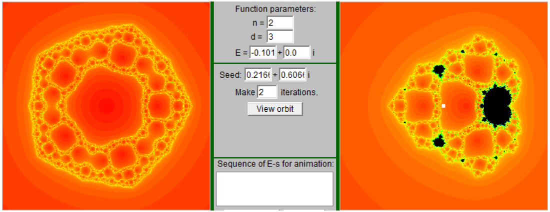

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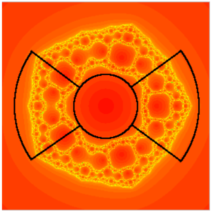
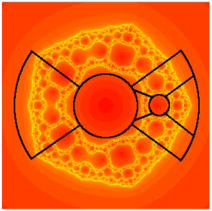


Bowties in bowties



- This is the preimage of the bowtie inside R^{λ}.

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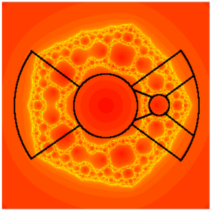

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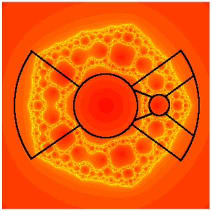

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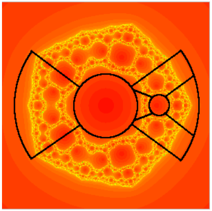

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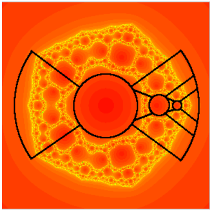
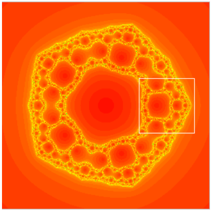
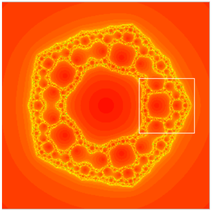
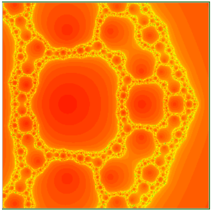


Zooming In

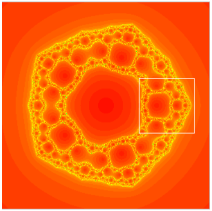


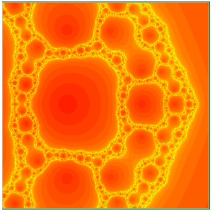
Zooming In

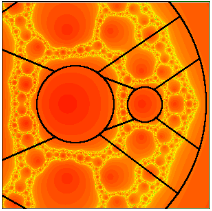




Zooming In







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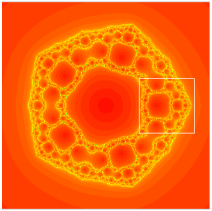
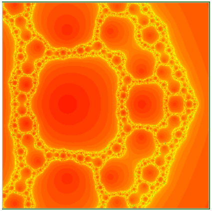
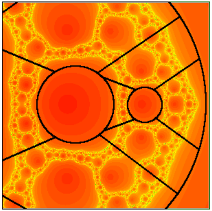
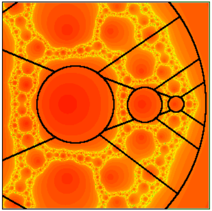
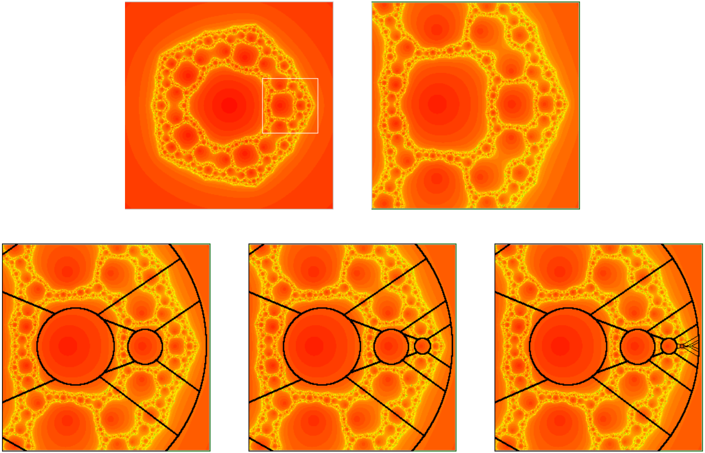




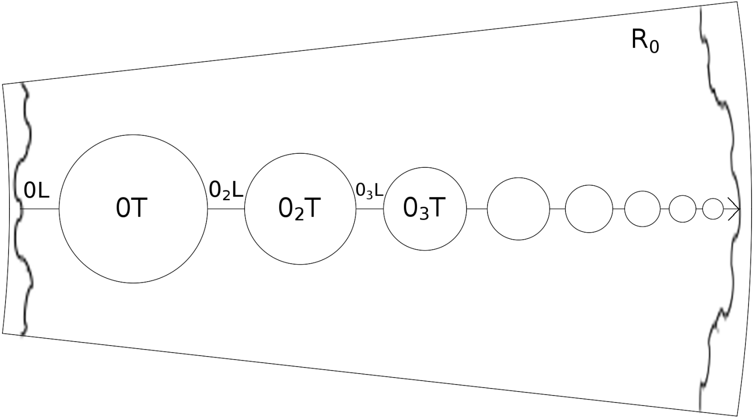




Zooming In



The Dynamical $\bar{0}TL$ Arc



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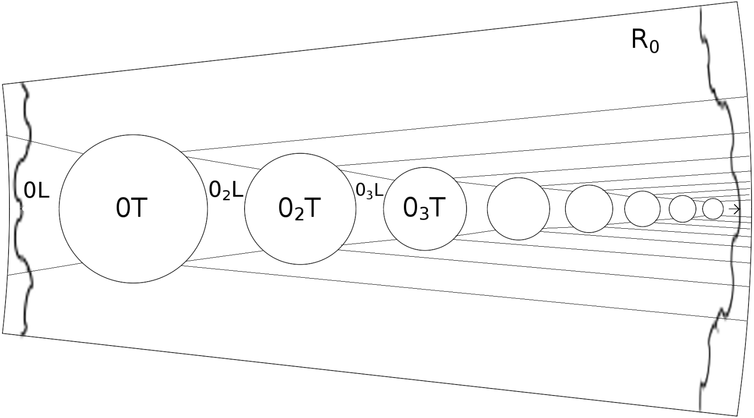

- This is a stylized representation of the $\bar{0}TL$ arc in the dynamical plane. With the wedges shown.

Dynamical TL Arc $\Longrightarrow$ Parameter SM Arc

- There is an arc of infinitely many alternating preimages of L^λ and $T_\mathcal{A}$ in R^λ in the dynamical plane.

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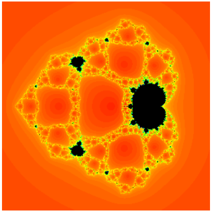
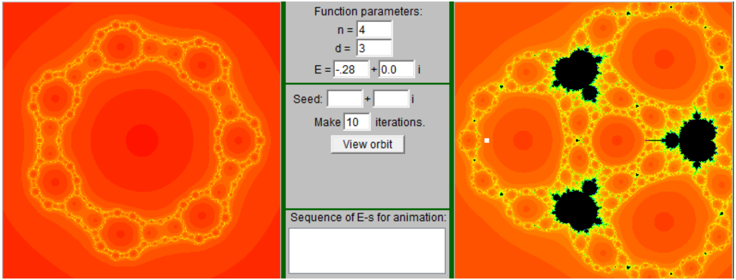


Outline

$n = 4, d = 3$


Function parameters:
n = 4
d = 3
E = -.28 + 0.0 i
Seed: + i
Make 10 iterations.
View orbit
Sequence of E-s for animation:

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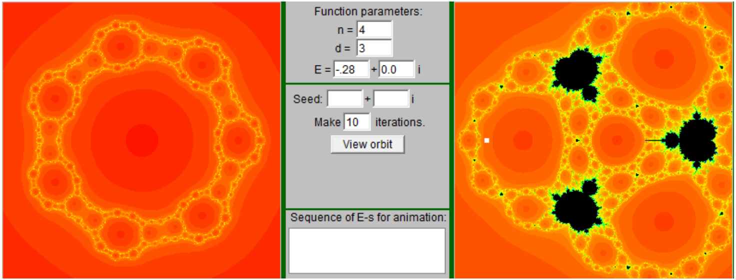

- The arc is still there.

Loptie

- The techniques still work, and we can do more.

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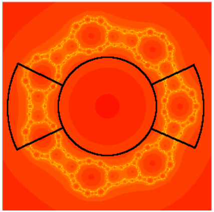

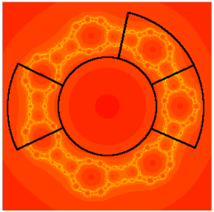

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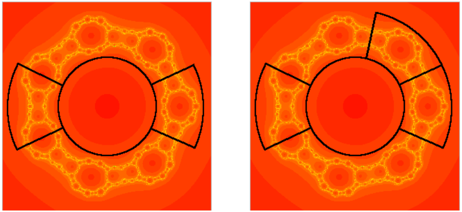

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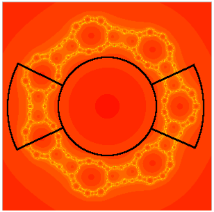

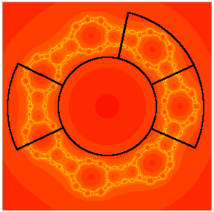

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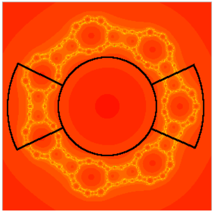

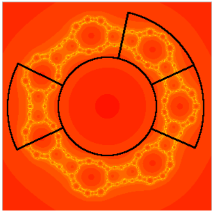

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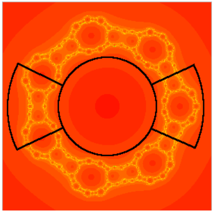

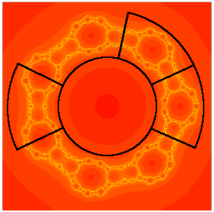

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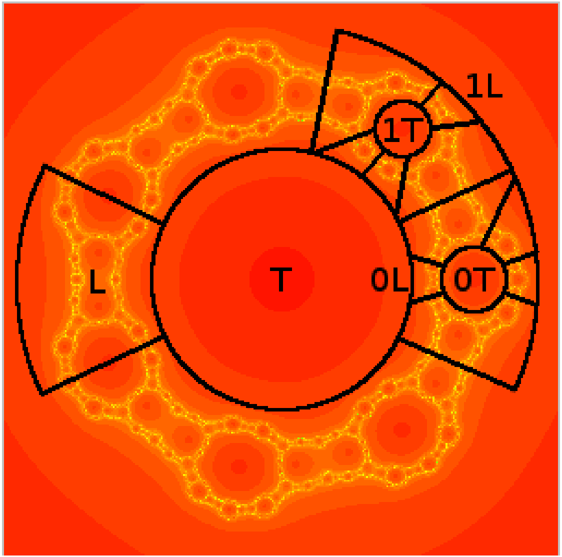
- R_0^λ still contains a preimage of the lopsided bowtie. R_1^λ also contains a preimage, but rotated.

Labeling



Scale is a Problem

- For this stylized R_0^λ, we label the preimages of the left wedge:

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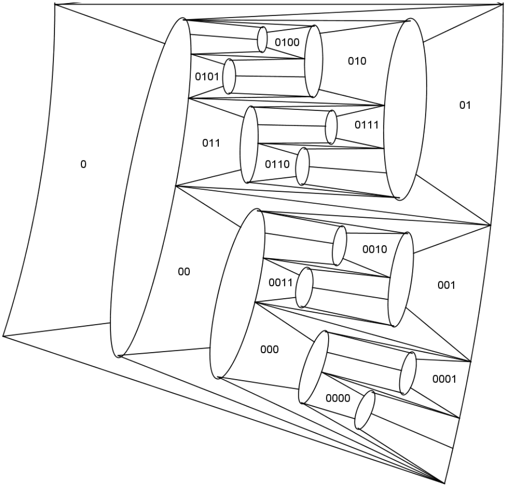
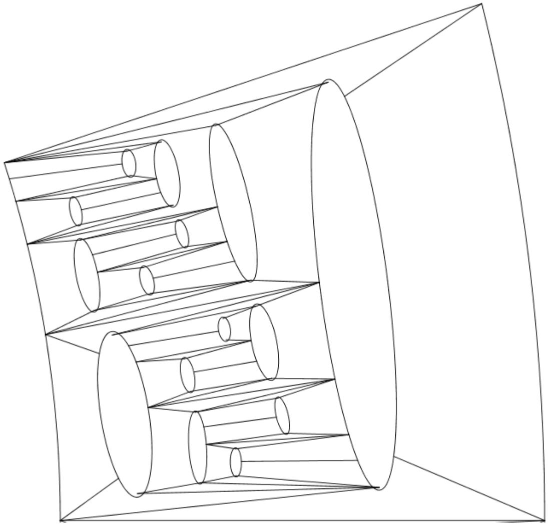
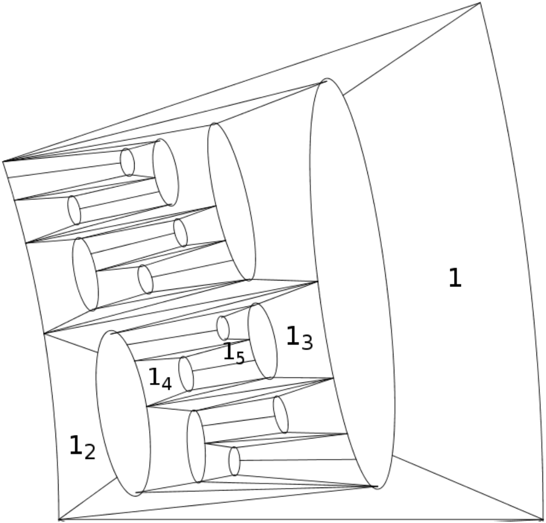


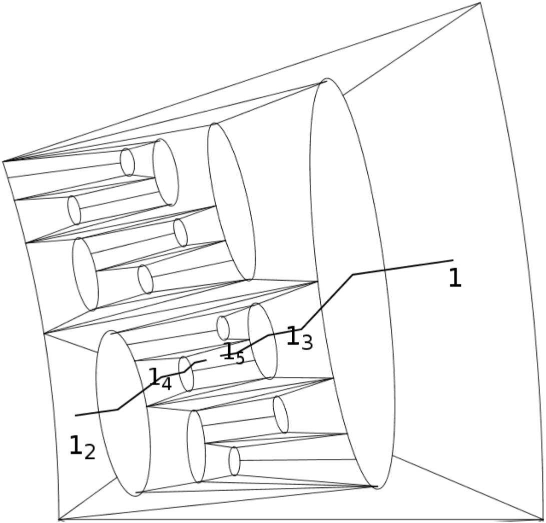
Looking for the Fixed Point in R_1^λ



Sequences of all $1's$



Arc of all $1'$s



Two Arcs

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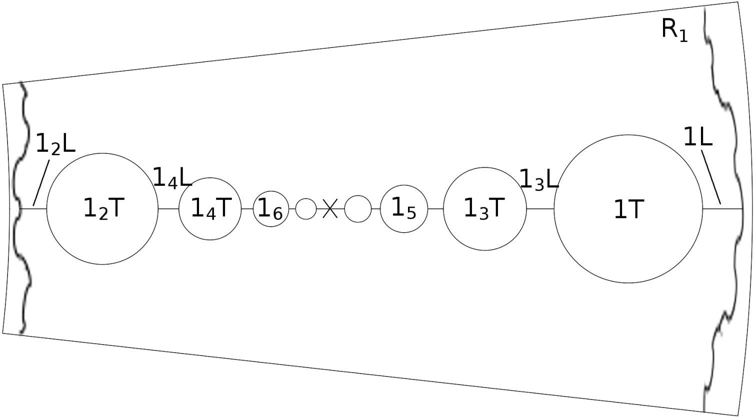
The Dynamical $\bar{1}TL$ Arc



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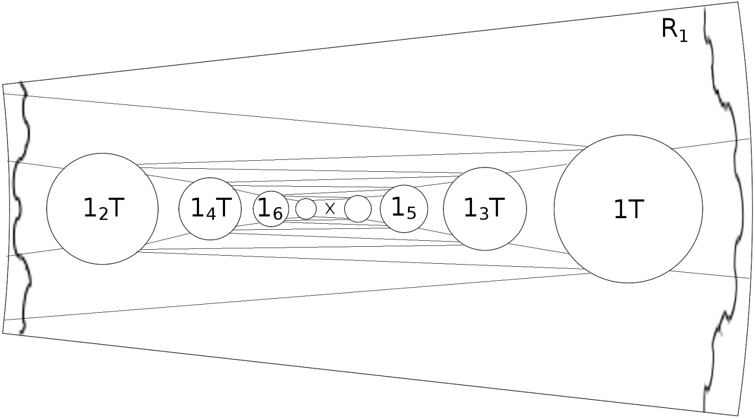

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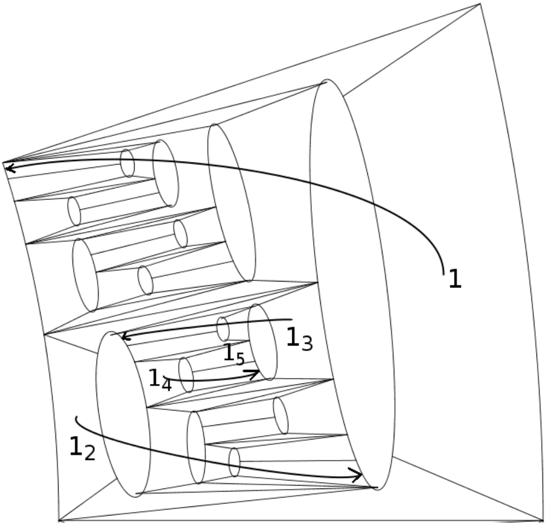
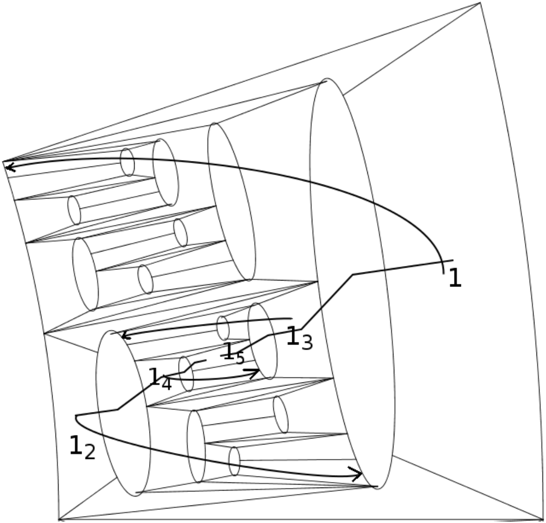
Outline

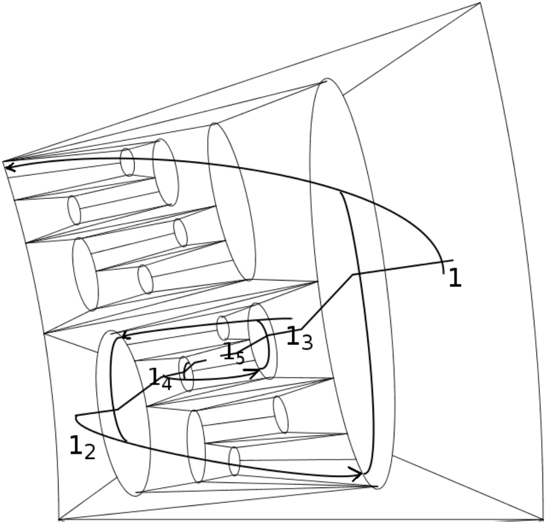
Infinitely many $\bar{0}TL$ arcs



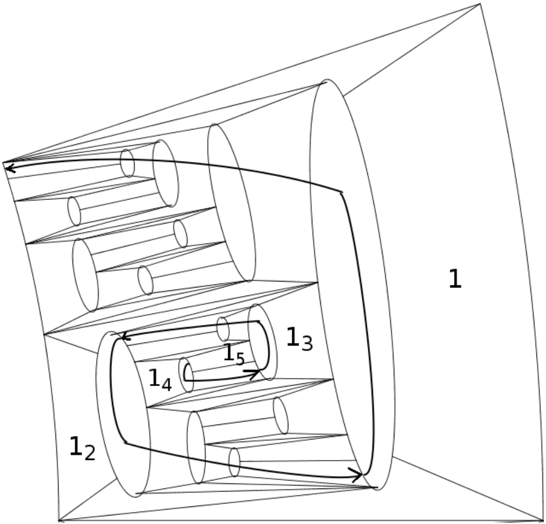
Infinitely Many $\bar{0}TL$ Arcs Intersecting the $\bar{1}TL$ Arc



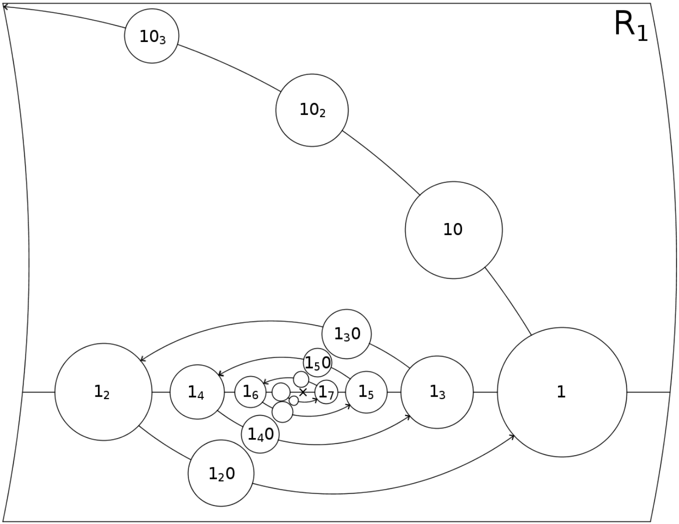
A Continuous Path for λ



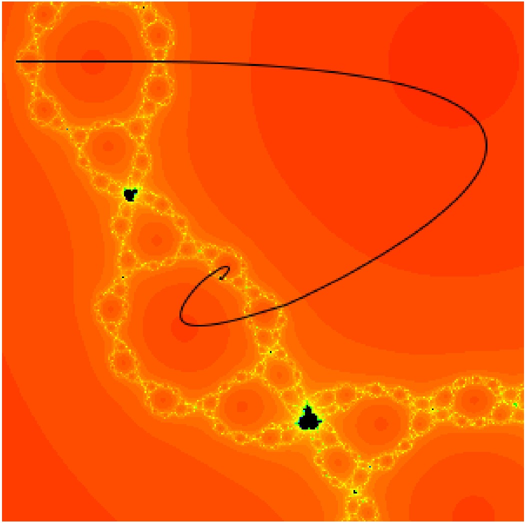
The $\bar{1}TL$ Spiral



Stylized $\overline{1}TL$ Spiral



The $0\overline{1}$ SM Spiral

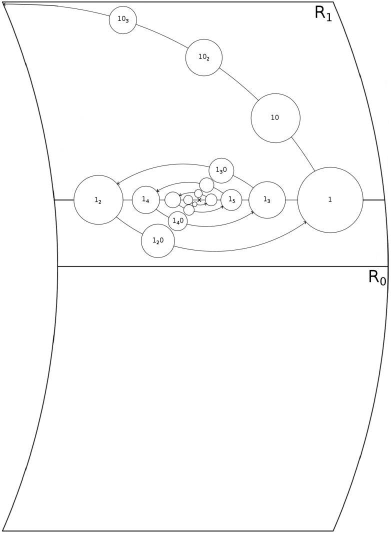


Mathematically Rigorous Statement

Theorem

There exists a $0\overline{1}$ SM arc below the negative real axis in the parameter plane that consists of infinitely many Mandelbrot sets $\mathcal{M}^k$ and infinitely many Sierpinski holes $\mathcal{E}^k$ both with $k \geq 3$. k denotes the base period of $\mathcal{M}^k$ and the escape time of $\mathcal{E}^k$.
Furthermore, there exists a $0\overline{1}$ SM spiral in the parameter plane that "spirals" from the Cantor set locus along infinitely many $\overline{0}$ type arcs while passing through each Sierpinski hole in the $0\overline{1}$ arc, and limits to λ such that $F_\lambda^2(c_0^\lambda)$ is the fixed point in R_1^λ.

Infinitely Many *TL* Spirals



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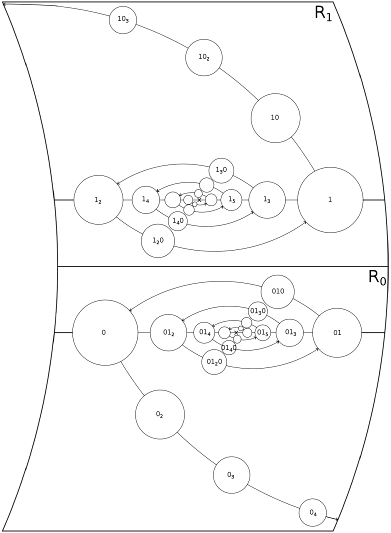

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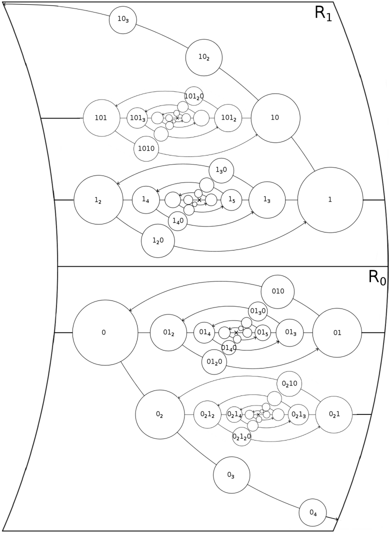

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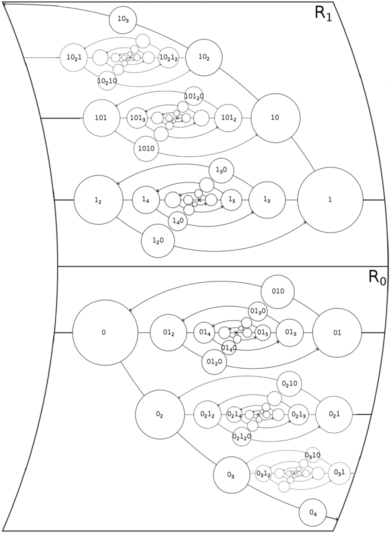
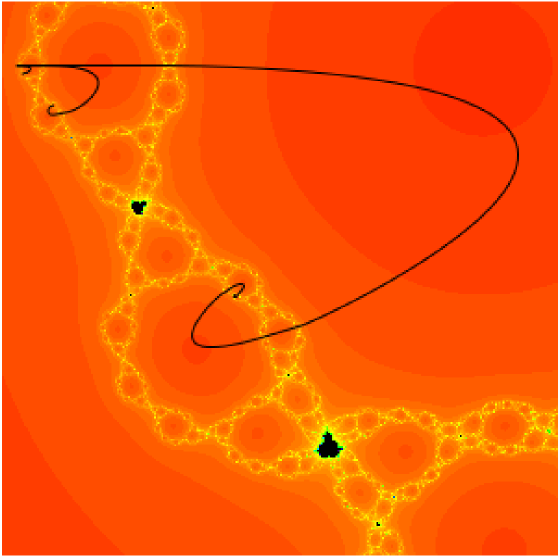
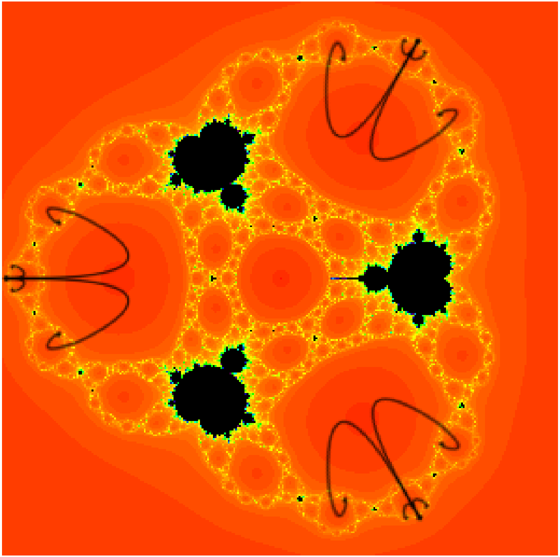


Infinitely Many SM Spirals



Infinitely Many SM Spirals x6



Future Directions

- Generalize to (n, d) such that $n \geq 4$ is even and $d \geq 3$ is odd.

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- Alternative upper right wedge $\implies$ alternative $\overline{2}$ arc $\implies$ alternative spiral for almost every exception.
- Alternative spirals exist for almost every (n, d), not only exceptional cases.

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- There is probably some way to prove the existence of a SM spiral for the $(6, 3), (4, 7)$ cases.
- We require n is even and d is odd, but there may be some way to tweak the argument to look at the case $n = d$ or n is odd, d is even.

The End

Thank you!

Outline

Equivalent Definitions of the Julia Set

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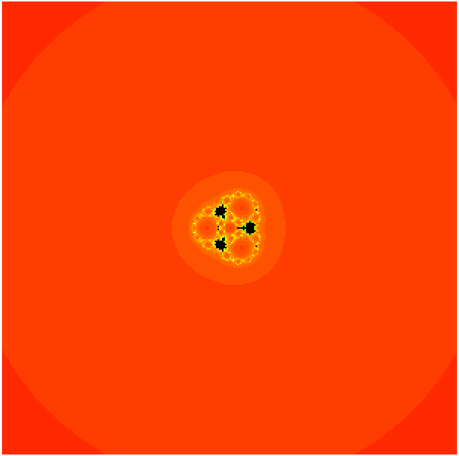
λ Annulus



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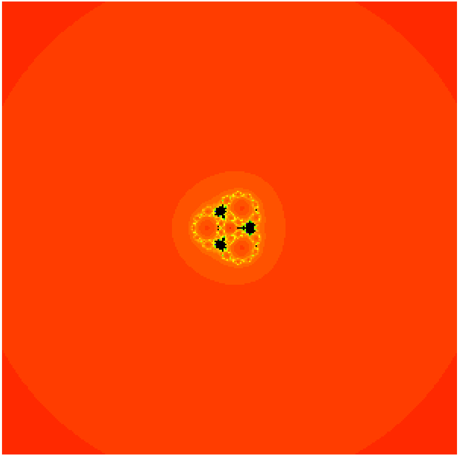

back

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- Then there exists a unique λ for which $v^\lambda = z_k^\lambda$,
- and that λ is the center of a Sierpinski hole with critical point escape time k.

$T_{\mathcal{A}}$ proves a Sierpinski hole

- We need a unique z_k^λ that varies analytically with λ and for which $F_\lambda^{k-1}(z_k^\lambda) = 0$.
- For that z_k^λ, we need a disk D in the parameter plane for which a critical value winds once around z_k^λ as λ winds once around the boundary of D.
- Then there exists a unique λ for which $v^\lambda = z_k^\lambda$,
- and that λ is the center of a Sierpinski hole with critical point escape time k.
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- As μ winds once around the boundary of $\mathcal{D}$, the critical value winds once around the region $V_\mu - U_\mu$. For such a family of polynomial-like maps, there exists a homeomorphic copy of the Mandelbrot set in $\mathcal{D}$.
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