Selection of dominant quasi-stationary states in 2D Navier-Stokes on the symmetric and asymmetric torus

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BU-Keio Workshop 2018

June 29, 2018

Introduction

Consider the incompressible 2D Navier-Stokes Equation with periodic boundaries on the domain $\mathcal{D}_\delta := [0, 2\pi\delta] \times [0, 2\pi]$

$$\begin{aligned} \partial_t \mathbf{u} = \nu\Delta\mathbf{u} &- (\mathbf{u}\cdot\nabla)\mathbf{u} - \nabla p. \\ \nabla\cdot\mathbf{u} &= 0 \end{aligned}$$

Recent numerical studies such as H.J.H. Clercx, D.C. Montgomery, and Z. Yin (2002) [4] and F. Bouchet and E. Simonnet, (2008) have shown certain families of functions to play a large role in the long time evolution of solutions. We call these **quasi-stationary**, or metastable solutions.

2D Vorticity Equation

These quasi-stationary solutions are defined via the vorticity,

$$\omega = (0, 0, 1) \cdot (\nabla \times \mathbf{u})$$

The 2D Vorticity equation is a scalar valued PDE, also on the domain $\mathcal{D}_\delta$ with periodic boundary conditions.

$$\partial_t \omega = \nu \Delta \omega - \mathbf{u} \cdot \nabla \omega, \qquad \mathbf{u} = \begin{pmatrix} \partial_y(-\Delta^{-1}) \\ -\partial_x(-\Delta^{-1}) \end{pmatrix} \omega$$

What is a metastable, or quasi-stationary solution?

Metastability can be thought of as a transient state a solution takes before the asymptotic limit is reached. We say "quasi-stationary" as these metastable states are rapidly attracting nearby solutions, just as a stationary solution would.

What is the asymptotic limit of the vorticity?

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}||w||_{L_2}^2 &= \int_D (ww_t)dxdy \\
&= \nu\int_D (w\Delta w)dxdy - \nu\int_D w(u\cdot\nabla w)dxdy \\
&= -\nu||\nabla w||_{L_2}^2 - \nu\int_D \nabla\cdot(w^2u)dxdy \\
&= -\nu||\nabla w||_{L_2}^2 - \nu\int_{\partial D} \vec{n}\cdot(w^2u) \\
&= -\nu||\nabla w||_{L_2}^2 \leq -\nu||w||_{L_2}^2
\end{aligned}
$$

Bar and Dipole States

Consider functions of the form

$$\omega(x,y) = e^{-\frac{\nu}{\delta^2}t}[a_1\cos(\frac{x}{\delta}) + a_2\sin(\frac{x}{\delta})] + e^{-\nu t}[a_3\cos(y) + a_4\sin(y)],$$

Certain members of this family have special names

- Bar states, or unidirectional flow, have the form

$$\omega_{bar}(x,t) = e^{-\frac{\nu}{\delta^2}t}\sin(x/\delta), \quad \omega_{bar}(y,t) = e^{-\nu t}\sin y$$

- Dipole states have the form

$$\omega_{dipole}(x,y,t) = e^{-\frac{\nu}{\delta^2}t}\sin(x/\delta) + e^{-\nu t}\sin y$$

Remark: When $\delta = 1$ both bars and dipoles are solutions but when $\delta \neq 1$ only bar states remain solutions

Contour Plots

These functional forms have been shown numerically to be quasi-stationary solutions to the 2D NS Vorticity equation

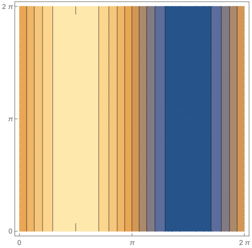

Figure: x-bar, $\omega = \sin(x/\delta)$

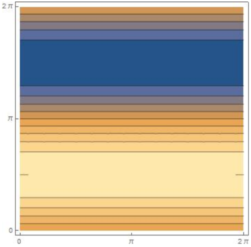

Figure: y-bar, $\omega = \sin(y)$

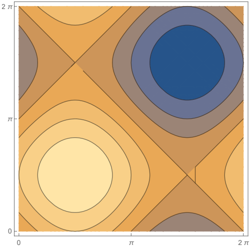

Figure: Dipole, $\sin(x/\delta) + \sin(y)$

Bouchet and Simonnet (2008): selection of dominant quasi-stationary depends on $\delta \approx 1$ [3]
Beck and Wayne (2012): Rapid convergence to bar [1] states.

Bars and dipoles in Fourier Space

There is a natural connection between these states and the Fourier modes of the vorticity.

$$\omega(x,y,t) = \sum_{\vec{k}\neq 0} \hat{\omega}(k_1,k_2)(t)e^{\mathrm{i}(k_1 x/\delta + k_2 y)}$$

We can measure how close the system is to a bar or dipole state via the relative energy in the lowest Fourier Modes, $\hat{\omega}(1,0)$ and $\hat{\omega}(0,1)$.

Define $R(t) := \dfrac{|\hat{\omega}(1,0)|^2}{|\hat{\omega}(0,1)|^2}$, and $Z(t) := \dfrac{|\hat{\omega}(1,0)|^2}{|\hat{\omega}(1,0)|^2 + |\hat{\omega}(0,1)|^2}$.

x-bar: $R(t) \to \infty \Leftrightarrow Z(t) \to 1$

y-bar: $R(t) \to 0 \Leftrightarrow Z(t) \to 0$

dipole: $R(t) \to r = \mathcal{O}(1) \Leftrightarrow Z(t) \to z \approx 1/2$

In Fourier Space, we now have an infinite dimensional system of ODE's for the Fourier Modes given by

$$\begin{aligned}\dot{\hat{\omega}}_{\vec{k}} &= -\frac{\nu}{\delta^2}|\vec{k}|_\delta^2\hat{\omega}_{\vec{k}} - \delta\sum_{\vec{l}}\frac{\langle\vec{k}^\perp,\vec{l}\rangle}{|\vec{l}|_\delta^2}\hat{\omega}_{\vec{k}-\vec{l}}\hat{\omega}_{\vec{l}} \\ &= -\frac{\nu}{\delta^2}|\vec{k}|_\delta^2\hat{\omega}_{\vec{k}} - \frac{\delta}{2}\sum_{\vec{j}+\vec{l}=\vec{k}}\langle\vec{j}^\perp,\vec{l}\rangle\left(\frac{1}{|\vec{l}|_\delta^2}-\frac{1}{|\vec{j}|_\delta^2}\right)\hat{\omega}_{\vec{j}}\hat{\omega}_{\vec{l}},\end{aligned}$$

where

$$\hat{\omega}_{\vec{k}} = \hat{\omega}(k_1,k_2), \qquad |\vec{k}|_\delta^2 = k_1^2+\delta^2k_2^2, \qquad \vec{k}^\perp = (k_2,-k_1)$$

Projection onto finite dimensional inertial manifold

Interested in the relative energy in the low modes. Project onto the lowest 8 Fourier modes.

$$\begin{aligned}
\omega_1 &:= \hat{\omega}(1,0), \qquad \omega_2 := \hat{\omega}(-1,0),\\
\omega_3 &:= \hat{\omega}(0,1), \qquad \omega_4 := \hat{\omega}(0,-1),\\
\omega_5 &:= \hat{\omega}(1,1), \qquad \omega_6 := \hat{\omega}(-1,1),\\
\omega_7 &:= \hat{\omega}(1,-1), \qquad \omega_8 := \hat{\omega}(-1,-1).
\end{aligned}$$

The variables $\omega_{1,2,3,4}$ correspond to the low modes, while $\omega_{5,6,7,8}$ represent the role of all the high modes. Note the following complex conjugacy relationship.

$$\omega_1 = \bar{\omega}_2, \quad \omega_3 = \bar{\omega}_4, \quad \omega_5 = \bar{\omega}_8, \quad \omega_7 = \bar{\omega}_8.$$

$$
\begin{aligned}
\dot{\omega}_1 &= -\frac{\nu}{\delta^2}\omega_1 + \frac{1}{\delta(1+\delta^2)}[\omega_3\omega_7 - \bar{\omega}_3\omega_5] \\
&\quad + \frac{3\delta^6}{2\nu(4+\delta^2)(1+\delta^2)^2}\omega_1(|\omega_5|^2 + |\omega_7|^2) \\
\dot{\omega}_3 &= -\nu\omega_3 + \frac{\delta^3}{(1+\delta^2)}[\bar{\omega}_1\omega_5 - \omega_1\bar{\omega}_7] \\
&\quad + \frac{3\delta^2}{2\nu(1+4\delta^2)(1+\delta^2)^2}\omega_3(|\omega_5|^2 + |\omega_7|^2) \\
\dot{\omega}_5 &= -\nu\frac{1+\delta^2}{\delta^2}\omega_5 - \frac{\delta^2-1}{\delta}\omega_1\omega_3 - \frac{\delta^6(3+\delta^2)}{2\nu(4+\delta^2)(1+\delta^2)}\omega_5|\omega_1|^2 \\
&\quad - \frac{1+3\delta^2}{2\nu\delta^2(1+4\delta^2)(1+\delta^2)}\omega_5|\omega_3|^2 \\
\dot{\omega}_7 &= -\nu\frac{1+\delta^2}{\delta^2}\omega_7 + \frac{\delta^2-1}{\delta}\omega_1\bar{\omega}_3 - \frac{\delta^6(3+\delta^2)}{2\nu(4+\delta^2)(1+\delta^2)}\omega_7|\omega_1|^2 \\
&\quad - \frac{1+3\delta^2}{2\nu\delta^2(1+4\delta^2)(1+\delta^2)}\omega_7|\omega_3|^2
\end{aligned}
$$

Low modes:

$$\dot{\omega}_1 = -\frac{\nu}{\delta^2}\omega_1 + \frac{1}{\delta(1+\delta^2)}[\omega_3\omega_7 - \bar{\omega}_3\omega_5] + \frac{3\delta^6}{2\nu(4+\delta^2)(1+\delta^2)^2}\omega_1(|\omega_5|^2 + |\omega_7|^2)$$

High modes:

$$\dot{\omega}_5 = -\nu\frac{1+\delta^2}{\delta^2}\omega_5 - \frac{\delta^2-1}{\delta}\omega_1\omega_3 - \frac{\delta^6(3+\delta^2)}{2\nu(4+\delta^2)(1+\delta^2)}\omega_5|\omega_1|^2 - \frac{1+3\delta^2}{2\nu\delta^2(1+4\delta^2)(1+\delta^2)}\omega_5|\omega_3|^2$$

$\delta = 1$ (Symmetric Domain)

In the symmetric case, our system simplifies a bit.

$$\begin{aligned}
\dot{\omega}_1 &= -\nu\omega_1 + \frac{1}{2}[\omega_3\omega_7 - \bar{\omega}_3\omega_5] + \frac{3}{40\nu}\omega_1(|\omega_5|^2 + |\omega_7|^2) \\
\dot{\omega}_3 &= -\nu\omega_3 + \frac{1}{2}[\bar{\omega}_1\omega_5 - \omega_1\bar{\omega}_7] + \frac{3}{40\nu}\omega_3(|\omega_5|^2 + |\omega_7|^2) \\
\dot{\omega}_5 &= -2\nu\omega_5 - \frac{1}{5\nu}\omega_5(|\omega_1|^2 + |\omega_3|^2) \\
\dot{\omega}_7 &= -2\nu\omega_7 - \frac{1}{5\nu}\omega_7(|\omega_1|^2 + |\omega_3|^2)
\end{aligned}$$

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Time scale separation

Lemma

Define $A(t) := |\omega_1(t)|^2 + |\omega_3(t)|^2$ *and* $B(t) := |\omega_5(t)|^2 + |\omega_7(t)|^2$*. Let* $t_0 = 1/\nu$, $\delta = 1$, *and denote the initial data by* $A(0) = A_0$ *and* $B(0) = B_0$*. We have*

$$A(t) + B(t) \leq (A_0 + B_0)e^{-2\nu t} \qquad \text{for all} \qquad t \geq 0.$$

Moreover, for all $0 \leq t \leq t_0$, $A(t) \geq A_0 e^{-2}$ *and* $B(t) \leq B_0 e^{-\frac{2A_0}{5\nu e^2}t}$*. Finally, for all* $t \geq t_0$, $B(t) \leq B_0 e^{-\frac{2A_0}{5\nu^2 e^2}}$*.*

Time scale separation

Proof.

The dynamics of A and B are governed by

$$\dot{A} = -2\nu A + \frac{3}{20\nu}AB$$
$$\dot{B} = -4\nu B - \frac{2}{5\nu}AB$$

The first claim follows from the fact that, since A and B are both nonnegative,

$$\frac{d}{dt}(A+B) = -2\nu(A+B) - 2\nu B - \frac{1}{4\nu}AB \leq -2\nu(A+B).$$

Furthermore $\dot{A} \geq -2\nu A \Rightarrow A(t) \geq A_0 e^{-2\nu t} \Rightarrow A(t) \geq A_0 e^{-2}$ for all $0 \leq t \leq t_0$. We then see that for all $0 \leq t \leq t_0$

$$\dot{B} \leq -\left(4\nu + \frac{2A_0}{5\nu e^2}\right) B \leq -\frac{2A_0}{5\nu e^2} B,$$

Main Deterministic Results Beck, C., Spiliopoulos (2017) [2]

The main results for the deterministic system can be summarized by the following:

- **Theorem 1.** For $\delta = 1$, both bar and dipole states exist as quasi-stationary states
- **Theorem 2.** For values of δ close to 1, if $\delta < 1$, y-bar states are the dominant quasi-stationary state (And for $\delta > 1$, x-bar states dominate).

The selection of metastable state depends on the relative size of the sides of the torus.

Viewing the problem as a perturbation

One approach is to view 2DNS on the asymmetric domain ($\delta \neq 1$) as a perturbation of the symmetric domain ($\delta = 1$)

Let $\delta = 1 + \epsilon_0 \epsilon$

Taylor expand, scale ω's, ν, and time appropriately ($\nu = \epsilon^{\alpha} \nu_0$, $t = \epsilon^{\beta} \tau, \omega_i = \epsilon^{\gamma_i} \Omega_i$) to reveal a slow fast system.

$$\frac{d}{d\tau}\Omega_1 = -\nu_0 \Omega_1 + h.o.t.$$
$$\frac{d}{d\tau}\Omega_3 = -\nu_0 \Omega_3 + h.o.t.$$
$$\frac{d}{d\tau}\Omega_5 = -\epsilon^{-1} \frac{1}{5\nu_0} \Omega_5 (|\Omega_1|^2 + |\Omega_3|^2) + h.o.t.$$
$$\frac{d}{d\tau}\Omega_7 = -\epsilon^{-1} \frac{1}{5\nu_0} \Omega_7 (|\Omega_1|^2 + |\Omega_3|^2) + h.o.t.$$

Dynamics away from the slow manifold

Assuming expansions of $\Omega_i(s)$ for $i = 1, 3, 5, 7$ to be of the form $\Omega_i(s) = \Omega_{i0}(s) + \epsilon\Omega_{i1}(s) + \mathcal{O}(\epsilon^2)$ away from the slow manifold, we find

$$\begin{aligned}
\Omega_{10} &= \Omega_{10}(0) \\
\Omega_{30} &= \Omega_{30}(0) \\
\Omega_{50} &= \Omega_{50}(0) e^{-\frac{|\Omega_{10}(0)|^2 + |\Omega_{30}(0)|^2}{5\nu_0} s} \\
\Omega_{70} &= \Omega_{70}(0) e^{-\frac{|\Omega_{10}(0)|^2 + |\Omega_{30}(0)|^2}{5\nu_0} s}.
\end{aligned}$$

. Here s is the fast time variable $s = \tau/\epsilon$

Dynamics on the slow manifold

Again assume solutions take the form $\Omega_i(\tau) = \Omega_{i0}(\tau) + \epsilon\Omega_{i1}(\tau) + \mathcal{O}(\epsilon^2)$. Then the dynamics on the slow manifold up to and including $\mathcal{O}(\epsilon)$ are given by

$$\begin{aligned}
\bar{\Omega}_1(\tau) &:= \Omega_{10}(0)e^{-\nu_0\tau} \\
&\quad + \epsilon\left(\Omega_{11}(0)e^{-\nu_0\tau} + \nu_0\epsilon_0\tau e^{-\nu_0\tau}\left[2\Omega_{10}(0) + K\right]\right) \\
\bar{\Omega}_3(\tau) &:= \Omega_{30}(0)e^{-\nu_0\tau} \\
&\quad + \epsilon\left(\Omega_{31}(0)e^{-\nu_0\tau} - \nu_0\epsilon_0 K\tau e^{-\nu_0\tau}\right).
\end{aligned}$$

Lemma

Let $0 < \epsilon \ll 1$*. Consider the approximations to* $|\Omega_1|^2$ *and* $|\Omega_3|^2$ *up to* $\mathcal{O}(\epsilon)$*. There exists positive times* τ_+ *and* τ_-*, for which, when* $\epsilon_0 = 1$*,* $\lim_{\tau \to \tau_+} \frac{|\bar{\Omega}_1|^2(\tau)}{|\bar{\Omega}_3|^2(\tau)} = \infty$*, indicating evolution to an x-bar state, and, when* $\epsilon_0 = -1$*,* $\lim_{\tau \to \tau_-} \frac{|\Omega_1|^2(\tau)}{|\Omega_1|^2(\tau)} = 0$*, indicating evolution to a y-bar state. The critical times* τ_+ *and* τ_- *are* $\mathcal{O}(1/\epsilon)$ *as* $\epsilon \to 0$*.*

Stochastic Forcing

$$\partial_t \omega = \nu \Delta \omega - \mathbf{u} \cdot \nabla \omega + \sqrt{2\nu} \partial_t \eta.$$

The noise, η_t, is a space time white noise of the form

$$\eta(t, x, y) = \sum_{k=(k_1,k_2)\in\mathbb{Z}^2} \sigma_k e^{\mathrm{i}(k_1 x/\delta + k_2 y)} W_k(t)$$

where $W(t) = (W_k(t))_k$ are i.i.d Weiner Processes.

Numerical studies have shown that in the presence of stochastic forcing, the asymptotic limit is not reached. Instead, the system may transition among the 3 quasi-stationary states.

Simulations

The figures below show the evolution of $Z_t = \frac{|\hat{\omega}(1,0)|^2}{|\hat{\omega}(1,0)|^2+|\hat{\omega}(0,1)|^2}$ when forcing is present in only the lowest modes $\omega_{\vec{k}}$ for $\vec{k} = (\pm 1, 0)$ or $\vec{k} = (0, \pm 1)$

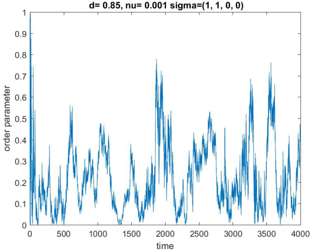

Figure: Transitions between y-bar and dipole for $\delta < 1$

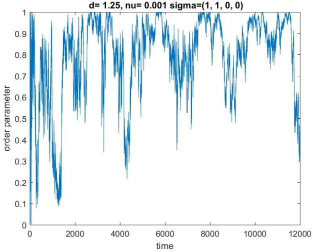

Figure: Transitions between x-bar and dipole for $\delta > 1$

Homogenization

With the stochastic forcing included, our finite dimensional model (after similar scaling) becomes

$$\omega_i' \quad = \quad b_i^\epsilon(\omega_{1,3,5,7};\nu,\sigma_{1,3,5,7}) + \Sigma_{ii}^\epsilon(\omega_{1,3,5,7};\nu,\sigma_{1,3,5,7})W_i'(\tau).$$

Where the drift b^ϵ is essentially the same as the vector field in the deterministic setting and diffusion matrix Σ^ϵ is given by

$$\Sigma^\epsilon = \begin{bmatrix} \sigma_1\sqrt{2\nu_0} & 0 & 0 & 0 \\ 0 & \sigma_3\sqrt{2\nu_0} & 0 & 0 \\ 0 & 0 & \epsilon^{-1}\sigma_5\sqrt{2\nu_0} & 0 \\ 0 & 0 & 0 & \epsilon^{-1}\sigma_7\sqrt{2\nu_0} \end{bmatrix}$$

We can use the backward Kolmogorov equation to view this stochastic problem from a PDE point of view.

$$\frac{\partial u^{\epsilon}}{\partial \tau} = \mathcal{L}u^{\epsilon} = b^{\epsilon} \cdot \nabla u^{\epsilon} + \frac{1}{2}(\Sigma^{\epsilon})^2 : \nabla\nabla u^{\epsilon}$$

If we use $u(\tau = 0) = \frac{\omega_1^2}{\omega_1^2+\omega_3^2}$ then $u(t) = E|\frac{\omega_1^2}{\omega_1^2+\omega_3^2}|$ as they evolve governed by the dynamics of the SDE. As in the deterministic case, we assume the solution u takes the form

$$u = u_0 + \epsilon u_1 + \epsilon^2 u_2 + \dots$$

We can plug this expansion into the backward Kolmogorov equation and match powers of ϵ to come up with PDE's for the individual $u_i's$. Existence of an invariant measure allows averaging of the fast variables. Our numerical simulations have shown what one would expect, namely the preference of an x bar state for $\epsilon_0 = 1$ and a preference for a y bar state for $\epsilon_0 = -1$

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