Selection of dominant quasi-stationary states in 2D Navier-Stokes on the symmetric and asymmetric torus

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BU-Keio Workshop 2018

June 29, 2018
Consider the incompressible 2D Navier-Stokes Equation with periodic boundaries on the domain $\mathcal{D}_\delta := [0, 2\pi \delta] \times [0, 2\pi]$

\[
\partial_t \mathbf{u} = \nu \Delta \mathbf{u} - (\mathbf{u} \cdot \nabla) \mathbf{u} - \nabla p.
\]

\[
\nabla \cdot \mathbf{u} = 0
\]

Recent numerical studies such as H.J.H. Clercx, D.C. Montgomery, and Z. Yin (2002) [4] and F. Bouchet and E. Simonnet, (2008) have shown certain families of functions to play a large role in the long time evolution of solutions. We call these quasi-stationary, or metastable solutions.
These quasi-stationary solutions are defined via the vorticity,

$$\omega = (0, 0, 1) \cdot (\nabla \times \mathbf{u})$$

The 2D Vorticity equation is a scalar valued PDE, also on the domain $D_\delta$ with periodic boundary conditions.

$$\partial_t \omega = \nu \Delta \omega - \mathbf{u} \cdot \nabla \omega,$$

$$\mathbf{u} = \left( \begin{array}{c} \partial_y (-\Delta^{-1}) \\ -\partial_x (-\Delta^{-1}) \end{array} \right) \omega$$
What is a metastable, or quasi-stationary solution?

Metastability can be thought of as a transient state a solution takes before the asymptotic limit is reached. We say “quasi-stationary” as these metastable states are rapidly attracting nearby solutions, just as a stationary solution would.

What is the asymptotic limit of the vorticity?

\[
\frac{1}{2} \frac{d}{dt} ||w||^2_{L_2} = \int_D (ww_t) dxdy
\]

\[
= \nu \int_D (w\Delta w) dxdy - \nu \int_D w(u \cdot \nabla w) dxdy
\]

\[
= -\nu ||\nabla w||^2_{L_2} - \nu \int_D \nabla \cdot (w^2 u) dxdy
\]

\[
= -\nu ||\nabla w||^2_{L_2} - \nu \int_{\partial D} \vec{n} \cdot (w^2 u)
\]

\[
= -\nu ||\nabla w||^2_{L_2} \leq -\nu ||w||^2_{L_2}
\]
Bar and Dipole States

Consider functions of the form

$$\omega(x, y) = e^{-\frac{\nu}{\delta^2} t} [a_1 \cos(\frac{x}{\delta}) + a_2 \sin(\frac{x}{\delta})] + e^{-\nu t} [a_3 \cos(y) + a_4 \sin(y)],$$

Certain members of this family have special names

- Bar states, or unidirectional flow, have the form
  $$\omega_{\text{bar}}(x, t) = e^{-\frac{\nu}{\delta^2} t} \sin(x/\delta), \quad \omega_{\text{bar}}(y, t) = e^{-\nu t} \sin y$$

- Dipole states have the form
  $$\omega_{\text{dipole}}(x, y, t) = e^{-\frac{\nu}{\delta^2} t} \sin(x/\delta) + e^{-\nu t} \sin y$$

**Remark:** When $\delta = 1$ both bars and dipoles are solutions but when $\delta \neq 1$ only bar states remain solutions
These functional forms have been shown numerically to be quasi-stationary solutions to the 2D NS Vorticity equation.

**Figure:** x-bar, \( \omega = \sin(x/\delta) \)

**Figure:** y-bar, \( \omega = \sin(y) \)

**Figure:** Dipole, \( \sin(x/\delta) + \sin(y) \)

**Bouchet and Simonnet (2008):** selection of dominant quasi-stationary depends on \( \delta \approx 1 \) [3]

Bars and dipoles in Fourier Space

There is a natural connection between these states and the Fourier modes of the vorticity.

$$\omega(x, y, t) = \sum_{\vec{k} \neq 0} \hat{\omega}(k_1, k_2)(t)e^{i(k_1 x/\delta + k_2 y)}$$

We can measure how close the system is to a bar or dipole state via the relative energy in the lowest Fourier Modes, $\hat{\omega}(1, 0)$ and $\hat{\omega}(0, 1)$.

Define $R(t) := \frac{|\hat{\omega}(1, 0)|^2}{|\hat{\omega}(0, 1)|^2}$, and $Z(t) := \frac{|\hat{\omega}(1, 0)|^2}{|\hat{\omega}(1, 0)|^2 + |\hat{\omega}(0, 1)|^2}$.

- x-bar: $R(t) \to \infty \iff Z(t) \to 1$
- y-bar: $R(t) \to 0 \iff Z(t) \to 0$
- dipole: $R(t) \to r = \mathcal{O}(1) \iff Z(t) \to z \approx 1/2$
In Fourier Space, we now have an infinite dimensional system of ODE’s for the Fourier Modes given by

\[ \dot{\hat{\omega}}_k = -\frac{\nu}{\delta^2} |\vec{k}|^2_\delta \hat{\omega}_k - \delta \sum_{\vec{l}} \frac{\langle \vec{k}^\perp, \vec{l} \rangle}{|\vec{l}|^2_\delta} \hat{\omega}_{\vec{k}-\vec{l}} \hat{\omega}_{\vec{l}} \]

\[ = -\frac{\nu}{\delta^2} |\vec{k}|^2_\delta \hat{\omega}_k - \frac{\delta}{2} \sum_{\vec{j}+\vec{l}=\vec{k}} \langle \vec{j}^\perp, \vec{l} \rangle \left( \frac{1}{|\vec{l}|^2_\delta} - \frac{1}{|\vec{j}|^2_\delta} \right) \hat{\omega}_{\vec{j}} \hat{\omega}_{\vec{l}}, \]

where

\[ \hat{\omega}_{\vec{k}} = \hat{\omega}(k_1, k_2), \quad |\vec{k}|^2_\delta = k_1^2 + \delta^2 k_2^2, \quad \vec{k}^\perp = (k_2, -k_1) \]
Projection onto finite dimensional inertial manifold

Interested in the relative energy in the low modes. Project onto the lowest 8 Fourier modes.

\[
\begin{align*}
\omega_1 & := \hat{\omega}(1, 0), & \omega_2 & := \hat{\omega}(-1, 0), \\
\omega_3 & := \hat{\omega}(0, 1), & \omega_4 & := \hat{\omega}(0, -1), \\
\omega_5 & := \hat{\omega}(1, 1), & \omega_6 & := \hat{\omega}(-1, 1), \\
\omega_7 & := \hat{\omega}(1, -1), & \omega_8 & := \hat{\omega}(-1, -1).
\end{align*}
\]

The variables \(\omega_1, 2, 3, 4\) correspond to the low modes, while \(\omega_5, 6, 7, 8\) represent the role of all the high modes. Note the following complex conjugacy relationship.

\[
\omega_1 = \overline{\omega}_2, \quad \omega_3 = \overline{\omega}_4, \quad \omega_5 = \overline{\omega}_8, \quad \omega_7 = \overline{\omega}_8.
\]
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\[ \dot{\omega}_1 = -\frac{\nu}{\delta^2} \omega_1 + \frac{1}{\delta(1 + \delta^2)} [\omega_3 \omega_7 - \bar{\omega}_3 \bar{\omega}_5] \]

\[ + \frac{3 \delta^6}{2 \nu(4 + \delta^2)(1 + \delta^2)^2} \omega_1 (|\omega_5|^2 + |\omega_7|^2) \]

\[ \dot{\omega}_3 = -\nu \omega_3 + \frac{\delta^3}{(1 + \delta^2)} [\bar{\omega}_1 \omega_5 - \omega_1 \bar{\omega}_7] \]

\[ + \frac{3 \delta^2}{2 \nu(1 + 4 \delta^2)(1 + \delta^2)^2} \omega_3 (|\omega_5|^2 + |\omega_7|^2) \]

\[ \dot{\omega}_5 = -\nu \frac{1 + \delta^2}{\delta^2} \omega_5 - \frac{\delta^2 - 1}{\delta} \omega_1 \omega_3 - \frac{\delta^6(3 + \delta^2)}{2 \nu(4 + \delta^2)(1 + \delta^2)^2} \omega_5 |\omega_1|^2 \]

\[- \frac{1 + 3 \delta^2}{2 \nu \delta^2(1 + 4 \delta^2)(1 + \delta^2)} \omega_5 |\omega_3|^2 \]

\[ \dot{\omega}_7 = -\nu \frac{1 + \delta^2}{\delta^2} \omega_7 + \frac{\delta^2 - 1}{\delta} \omega_1 \bar{\omega}_3 - \frac{\delta^6(3 + \delta^2)}{2 \nu(4 + \delta^2)(1 + \delta^2)} \omega_7 |\omega_1|^2 \]

\[- \frac{1 + 3 \delta^2}{2 \nu \delta^2(1 + 4 \delta^2)(1 + \delta^2)} \omega_7 |\omega_3|^2 \]
Low modes:

\[ \dot{\omega}_1 = -\frac{\nu}{\delta^2} \omega_1 + \frac{1}{\delta(1 + \delta^2)} [\omega_3 \omega_7 - \bar{\omega}_3 \omega_5] \]

\[ + \frac{3\delta^6}{2\nu(4 + \delta^2)(1 + \delta^2)^2} \omega_1 (|\omega_5|^2 + |\omega_7|^2) \]

High modes:

\[ \dot{\omega}_5 = -\nu \frac{1 + \delta^2}{\delta^2} \omega_5 - \frac{\delta^2 - 1}{\delta} \omega_1 \omega_3 - \frac{\delta^6 (3 + \delta^2)}{2\nu(4 + \delta^2)(1 + \delta^2)} \omega_5 |\omega_1|^2 \]

\[-\frac{1 + 3\delta^2}{2\nu \delta^2 (1 + 4\delta^2)(1 + \delta^2)} \omega_5 |\omega_3|^2 \]
\( \delta = 1 \) (Symmetric Domain)

In the symmetric case, our system simplifies a bit.

\[
\begin{align*}
\dot{\omega}_1 &= -\nu \omega_1 + \frac{1}{2} [\omega_3 \omega_7 - \bar{\omega}_3 \omega_5] + \frac{3}{40 \nu} \omega_1 (|\omega_5|^2 + |\omega_7|^2) \\
\dot{\omega}_3 &= -\nu \omega_3 + \frac{1}{2} [\bar{\omega}_1 \omega_5 - \omega_1 \bar{\omega}_7] + \frac{3}{40 \nu} \omega_3 (|\omega_5|^2 + |\omega_7|^2) \\
\dot{\omega}_5 &= -2\nu \omega_5 - \frac{1}{5 \nu} \omega_5 (|\omega_1|^2 + |\omega_3|^2) \\
\dot{\omega}_7 &= -2\nu \omega_7 - \frac{1}{5 \nu} \omega_7 (|\omega_1|^2 + |\omega_3|^2)
\end{align*}
\]
Time scale separation

**Lemma**

Define $A(t) := |\omega_1(t)|^2 + |\omega_3(t)|^2$ and $B(t) := |\omega_5(t)|^2 + |\omega_7(t)|^2$. Let $t_0 = 1/\nu$, $\delta = 1$, and denote the initial data by $A(0) = A_0$ and $B(0) = B_0$. We have

$$A(t) + B(t) \leq (A_0 + B_0)e^{-2\nu t} \quad \text{for all} \quad t \geq 0.$$ 

Moreover, for all $0 \leq t \leq t_0$, $A(t) \geq A_0e^{-2}$ and $B(t) \leq B_0e^{-\frac{2A_0}{5\nu e^2} t}$. Finally, for all $t \geq t_0$, $B(t) \leq B_0e^{-\frac{2A_0}{5\nu^2 e^2}}$. 


Time scale separation

Proof.

The dynamics of $A$ and $B$ are governed by

\[
\dot{A} = -2\nu A + \frac{3}{20\nu} AB
\]
\[
\dot{B} = -4\nu B - \frac{2}{5\nu} AB
\]

The first claim follows from the fact that, since $A$ and $B$ are both nonnegative,

\[
\frac{d}{dt}(A + B) = -2\nu(A + B) - 2\nu B - \frac{1}{4\nu} AB \leq -2\nu(A + B).
\]

Furthermore $\dot{A} \geq -2\nu A \Rightarrow A(t) \geq A_0 e^{-2\nu t} \Rightarrow A(t) \geq A_0 e^{-2t}$ for all $0 \leq t \leq t_0$. We then see that for all $0 \leq t \leq t_0$

\[
\dot{B} \leq - \left(4\nu + \frac{2A_0}{5\nu e^2}\right) B \leq - \frac{2A_0}{5\nu e^2} B,
\]
The main results for the deterministic system can be summarized by the following:

- **Theorem 1.** For $\delta = 1$, both bar and dipole states exist as quasi-stationary states
- **Theorem 2.** For values of $\delta$ close to 1, if $\delta < 1$, $y$-bar states are the dominant quasi-stationary state (And for $\delta > 1$, $x$-bar states dominate).

The selection of metastable state depends on the relative size of the sides of the torus.
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**Viewing the problem as a perturbation**

One approach is to view 2DNS on the asymmetric domain ($\delta \neq 1$) as a perturbation of the symmetric domain ($\delta = 1$)

Let $\delta = 1 + \epsilon_0 \epsilon$

Taylor expand, scale $\omega$’s, $\nu$, and time appropriately ($\nu = \epsilon^\alpha \nu_0$, $t = \epsilon^\beta \tau$, $\omega_i = \epsilon^{\gamma_i} \Omega_i$) to reveal a slow fast system.

\[
\begin{align*}
\frac{d}{d\tau} \Omega_1 &= -\nu_0 \Omega_1 + h.o.t. \\
\frac{d}{d\tau} \Omega_3 &= -\nu_0 \Omega_3 + h.o.t. \\
\frac{d}{d\tau} \Omega_5 &= -\epsilon^{-1} \frac{1}{5\nu_0} \Omega_5 (|\Omega_1|^2 + |\Omega_3|^2) + h.o.t. \\
\frac{d}{d\tau} \Omega_7 &= -\epsilon^{-1} \frac{1}{5\nu_0} \Omega_7 (|\Omega_1|^2 + |\Omega_3|^2) + h.o.t.
\end{align*}
\]
Dynamics away from the slow manifold

Assuming expansions of \( \Omega_i(s) \) for \( i = 1, 3, 5, 7 \) to be of the form
\[
\Omega_i(s) = \Omega_{i0}(s) + \epsilon \Omega_{i1}(s) + O(\epsilon^2)
\]
away from the slow manifold, we find

\[
\begin{align*}
\Omega_{10} &= \Omega_{10}(0) \\
\Omega_{30} &= \Omega_{30}(0) \\
\Omega_{50} &= \Omega_{50}(0)e^{-\frac{|\Omega_{10}(0)|^2 + |\Omega_{30}(0)|^2}{5\nu_0}s} \\
\Omega_{70} &= \Omega_{70}(0)e^{-\frac{|\Omega_{10}(0)|^2 + |\Omega_{30}(0)|^2}{5\nu_0}s}.
\end{align*}
\]

Here \( s \) is the fast time variable \( s = \tau/\epsilon \).
Dynamics on the slow manifold

Again assume solutions take the form
\[ \Omega_i(\tau) = \Omega_{i0}(\tau) + \epsilon \Omega_{i1}(\tau) + O(\epsilon^2). \]
Then the dynamics on the slow manifold up to and including \( O(\epsilon) \) are given by

\[ \bar{\Omega}_1(\tau) := \Omega_{10}(0)e^{-\nu_0\tau} \]
\[ + \epsilon \left( \Omega_{11}(0)e^{-\nu_0\tau} + \nu_0\epsilon_0\tau e^{-\nu_0\tau} \left[ 2\Omega_{10}(0) + K \right] \right) \]

\[ \bar{\Omega}_3(\tau) := \Omega_{30}(0)e^{-\nu_0\tau} \]
\[ + \epsilon \left( \Omega_{31}(0)e^{-\nu_0\tau} - \nu_0\epsilon_0 K \tau e^{-\nu_0\tau} \right) . \]
Lemma

Let $0 < \epsilon \ll 1$. Consider the approximations to $|\Omega_1|^2$ and $|\Omega_3|^2$ up to $O(\epsilon)$. There exists positive times $\tau_+$ and $\tau_-$, for which, when $\epsilon_0 = 1$,

$$\lim_{\tau \to \tau_+} \frac{|\Omega_1|^2(\tau)}{|\Omega_3|^2(\tau)} = \infty,$$

indicating evolution to an $x$-bar state, and, when $\epsilon_0 = -1$,

$$\lim_{\tau \to \tau_-} \frac{|\Omega_1|^2(\tau)}{|\Omega_1|^2(\tau)} = 0,$$

indicating evolution to a $y$-bar state.

The critical times $\tau_+$ and $\tau_-$ are $O(1/\epsilon)$ as $\epsilon \to 0$. 


Stochastic Forcing

\[ \partial_t \omega = \nu \Delta \omega - \mathbf{u} \cdot \nabla \omega + \sqrt{2\nu} \partial_t \eta. \]

The noise, \( \eta_t \), is a space time white noise of the form

\[ \eta(t, x, y) = \sum_{k \in \mathbb{Z}^2} \sigma_k e^{i(k_1 x/\delta+k_2 y)} W_k(t) \]

where \( W(t) = (W_k(t))_k \) are i.i.d Weiner Processes.

Numerical studies have shown that in the presence of stochastic forcing, the asymptotic limit is not reached. Instead, the system may transition among the 3 quasi-stationary states.
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Simulations

The figures below show the evolution of $Z_t = \frac{|\hat{\omega}(1,0)|^2}{|\hat{\omega}(1,0)|^2 + |\hat{\omega}(0,1)|^2}$ when forcing is present in only the lowest modes $\omega_{\vec{k}}$ for $\vec{k} = (\pm 1, 0)$ or $\vec{k} = (0, \pm 1)$.

![Figure: Transitions between y-bar and dipole for $\delta < 1$](image1)

![Figure: Transitions between x-bar and dipole for $\delta > 1$](image2)
Homogenization

With the stochastic forcing included, our finite dimensional model (after similar scaling) becomes

$$\omega'_i = b^\epsilon_i(\omega_{1,3,5,7}; \nu, \sigma_{1,3,5,7}) + \sum^\epsilon_{ii}(\omega_{1,3,5,7}; \nu, \sigma_{1,3,5,7})W_i'(\tau).$$

Where the drift $b^\epsilon$ is essentially the same as the vector field in the deterministic setting and diffusion matrix $\sum^\epsilon$ is given by

$$\sum^\epsilon = \begin{bmatrix}
\sigma_1 \sqrt{2\nu_0} & 0 & 0 & 0 \\
0 & \sigma_3 \sqrt{2\nu_0} & 0 & 0 \\
0 & 0 & \epsilon^{-1} \sigma_5 \sqrt{2\nu_0} & 0 \\
0 & 0 & 0 & \epsilon^{-1} \sigma_7 \sqrt{2\nu_0}
\end{bmatrix}$$
We can use the backward Kolmogorov equation to view this stochastic problem from a PDE point of view.

\[
\frac{\partial u^\epsilon}{\partial \tau} = \mathcal{L}u^\epsilon = b^\epsilon \cdot \nabla u^\epsilon + \frac{1}{2}(\Sigma^\epsilon)^2 : \nabla \nabla u^\epsilon
\]

If we use \( u(\tau = 0) = \frac{\omega_1^2}{\omega_1^2 + \omega_3^2} \) then \( u(t) = E\left| \frac{\omega_1^2}{\omega_1^2 + \omega_3^2} \right| \) as they evolve governed by the dynamics of the SDE. As in the deterministic case, we assume the solution \( u \) takes the form

\[
u = u_0 + \epsilon u_1 + \epsilon^2 u_2 + \ldots
\]

We can plug this expansion into the backward Kolmogorov equation and match powers of \( \epsilon \) to come up with PDE’s for the individual \( u_i \)'s. Existence of an invariant measure allows averaging of the fast variables. Our numerical simulations have shown what one would expect, namely the preference of an x bar state for \( \epsilon_0 = 1 \) and a preference for a y bar state for \( \epsilon_0 = -1 \).
References

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