RILEY GROUPS AND CARUSO SEMIGROUPS

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For $\beta \in \mathbb{C}^*$ we study the Caruso semigroup $S_\beta$ generated by the Mobius transformations

$$f_\beta(z) = \beta + \frac{1}{z}, \quad g_\beta(z) = -\beta + \frac{1}{z}$$

acting on the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \infty$.

Motivation: each random sequence with $Z_{n+1} = Z_{n-1} \pm \beta Z_n$ determines a semigroup orbit

$$r_{n+1} = \pm \beta + r_n,$$

where $r_n = Z_{n-1}/Z_n$.

Q: For what $\beta \in \mathbb{C}^* \setminus \{0\}$ does $S_\beta$ have a thick attractor $A_\beta$? (thick means the basin of $A_\beta$ is a neighborhood of $A_\beta$)

By our paper in ETDS 2012, $\beta$ is equivalent to:

For what $\beta$ are the Julia sets $J_\beta = J(S_\beta)$ and $J_\beta' = J(S_\beta')$ disjoint?

For such $\beta$, $A_\beta = J_\beta'$ is a thick attractor for $S_\beta$ (and $J_\beta$ is a thick attractor for $S_\beta'$).

$A_\beta$ exists for $|\beta| > 2$ since for such $\beta$

$$|Z| \geq 1 \Rightarrow |\pm \beta + \frac{1}{Z}| \geq k > 1,$$

where $k = |\beta|-1$.

Moreover, $\beta + D, -\beta + D$ are disjoint (where $D$ is the unit disc) so $A_\beta$ has a trivial symbolic dynamics: $A_\beta \cong \{0,1,3\}$

where $f, g$ act respectively by $i_z, i_z$, where

$$i_z(z_0, z_1, ...) = (0, z_0, z_1, ...) \quad \text{and} \quad i_z(z_0, z_1, ...) = (1, z_0, z_1, ...)$$.
Using the theory of Riley groups we'll extend this result to an open annular region $\mathcal{A} \subset \mathbb{C}$ (the Koebe slice) and explore the remaining cases $\beta \in \mathcal{A} \setminus \mathcal{I}$ and $\beta \in \mathcal{E} \setminus \mathcal{I}$.

A Riley group is a Moebius group generated by 2 parabolic elements $z + a$ (fixing $\infty$) and $z/(bz + 1)$ (fixing 0), $a \neq 0 \pm b$.

Especially the group

$$P_\beta = \langle t_\beta, u_\beta \rangle,$$

$P_\beta$ is closed related to the group $G_\beta$ generated by $S_\beta$, namely the even-length words in $t_\beta^\pm$, $u_\beta^\pm$ are the even-length words in $f_\beta^\pm$, $g_\beta^\pm$ (for instance $f_\beta^2 = t_\beta u_\beta$, $f_\beta g_\beta = t_\beta u_\beta^{-1}$) so $P_\beta$, $G_\beta$ are commensurable. As $P_\beta$ is non-elementary, these groups have the same ordinary set $\Omega_\beta$ (defined as the largest invariant open set where $P_\beta$ acts discretely).

When $\Omega_\beta$ is nonempty we get an orbifold $\Sigma_\beta = \Omega_\beta / P_\beta$.

$|\beta| > 2$, let $L$ be the line $Re(\frac{z}{\beta}) = \frac{1}{2}$. The circles $\pm 1/L$ and the lines $\pm L$ bound a fundamental domain for $\Sigma_\beta$, which contains the unit circle $S^1$.

Drawing shows $\Sigma_\beta$ has genus 0 and 4 punctures.

The slice $\mathcal{A}$ is defined by: $\beta \in \mathcal{A}$ whenever $t_\beta$, $u_\beta$ act freely on $\Omega_\beta$ so that $\Sigma_\beta$ has genus 0 and 4 punctures.
One knows that $\mathbb{S}$ is open in $\mathbb{C}^*$ and homeomorphic to $S^1 \times \mathbb{R}$. Moreover $\mathbb{L} = \mathbb{S}$ is a Jordan curve (Okkoaka - Miyachi, 2008). We have a partial answer to $Q$.

**Theorem:** $A_\beta$ exists for $\beta \in \mathbb{S}$ and does not exist for $\beta \notin \mathbb{S}$.

**Sketch of Proof:** For $|\beta| > 2$ define $E_\beta$ as the geodesic in $\Sigma_\beta$ (for the Poincaré metric) isotopic to $S^1$. For $\beta \in \mathbb{S}$ define the geodesic $E = E_\beta < \Sigma_\beta$ by continuation from $|\beta| > 2$.

One checks that $E$ is a simple closed geodesic separating $0$ from $\infty$, $f_\pm + i \mathbb{R} \subset E = -E = 1/E$, and the $p(E)$, $p \in P_\beta$, are disjoint.

Let $N = N_\beta$ be the closed neighborhood of $\infty$ bounded by $E_\beta$. $E$ is disjoint from $E \pm \beta$ so $f_\beta(N)$, $g_\beta(N) = \beta + 1/N$, $-\beta + 1/N \subset \text{Int}(N)$.

This $N$ is an $S_\beta$-block (as defined in our paper, op. cit.) so $A_\beta$ exists. (Furthermore $\pm \beta + 1/N$ are disjoint so $S_\beta$ has trivial symbolic dynamics.)

Note: another proof uses that the groups $P_\beta$, $\beta \in \mathbb{S}$, are quasi-conformally conjugate, which implies that all the $\Lambda_{A_\beta} \overset{\text{def}}{=} \Lambda(P_\beta) = \Lambda(G_\beta)$ are Cantor sets, likewise the $S_\beta$ and $S'_\beta = A_\beta$ are Cantor sets.

The case $\beta \in \mathbb{S}$ uses certain cusps $c \in \mathbb{S}$ for which $P_c$ has an additional parabolic (that is a parabolic element not conjugate to one of the parabolic generators). Such a cusp $c$ is the endpoint of a pleating ray in $\mathbb{L}$, corresponding to a particular class of simple closed geodesics in $\Sigma_\beta$ that degenerates...
as $\beta \to c$ along the ray (in the sense that the length of the geodesic approaches zero). This geodesic is characterized by a rational number modulo $2\pi/q$ in $\mathbb{Q}/\mathbb{Z}$, where the geodesic cuts $E_p$ $q > 0$ times and winds $p$ times (in an appropriate sense). Pleating rays were studied by Keen and Series and the corresponding cusp groups $P_c$ were studied by Wright. Wright finds $2q$ discs in $\hat{C}$ that are cyclically arranged and tangent in successive pairs, with 2 additional tangencies at 0 and at $\infty$. (For $p/q = 1/3$, this configuration is

![Diagram](image)

where the $2q = 6$ $x$'s mark the successive tangencies.) For $\beta$ near $c$ on the pleating ray there are $2q$ slightly overlapping discs whose union contains $E_p$. With this in mind, we define $E_c = E_c^\perp / E_c$ and we define $N_c$ as before. One checks that $J_c \subset N_c$ but $J_c \cap E_c$ is the finite set of $x$'s. This set is also $J_c \cap E_c^\perp$. Hence $P_c$ has no strong attractor. As the existence of a strong attractor is a stable property and as the cusps $c$ are known to be dense in $E$, we see that $A_3$ does not exist for $BG \in \mathbb{C}$.

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This theorem exploits the fact that the semigroups $S_\beta$, $\beta \in \mathcal{E}$, contains representatives for the simple closed geodesics of interest in $\Sigma_\mathcal{E}$.

While all the $P_\beta$, $\beta \in \mathcal{E}$, are alike there is a dichotomy for the $P_\beta$, $\beta \in \mathcal{E}$. All these $P_\beta$ are discrete groups and free on the generators $t_\beta, \rho_\beta$ but their limit sets vary drastically. For $\beta = c$ a cusp, $\Lambda_c$ is a gasket (that is the ordinary set $\Omega_c$ is a union of disjoint discs, including the $2g$ discs considered by Wright) and $P_c$ is geometrically finite. For $\beta$ not a cusp, however, $\Lambda_\beta = \hat{\mathcal{C}}$ (so $\Omega_\beta$ is empty) and $P_\beta$ is geometrically infinite. One expects a similar dichotomy for the semigroups $S_\beta$, namely that $J_\beta \cap J'_\beta$ is infinite when $\beta \in \mathcal{E}$ is not a cusp.

The $P_\beta$ for $\beta \in \mathcal{E}$ vary wildly. Only countably many are discrete but these include many interesting examples associated to 2-bridge link complements. As far as we know, $J_\beta = \Lambda_\beta = J'_\beta$ for $\beta \in \mathcal{E}$. This is known when $[\text{id} \in S_\beta]$ (especially when $S_\beta$ contains a torsion element). This condition on $\beta$ may be denoted $\mathcal{E}'$, in which case there are no $A_\beta$, $\beta \in \mathcal{E}'$. 