RILEY GROUPS AND CARUSO SEMIGROUPS

David Fried (with Sebastian Marotta and Rich Stankewitz)

For $\beta \in \mathbb{C}^*$ we study the Caruso semigroup S_β generated by the Mobius transformations

$$f_\beta(z) = \beta + \frac{1}{z}, \quad g_\beta(z) = -\beta + \frac{1}{z},$$

acting on the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \infty$.

Motivation: each random (nonzero) sequence with $z_{n+1} = z_{n-1} \pm \beta z_n$ determines a semigroup orbit

$$r_{n+1} = \pm\beta + r_n,$$

where $r_n = z_{n-1}/z_n$

Q For what $\beta \in \mathbb{C}^* = \mathbb{C} - 0$ does S_β have a thick attractor A_β?
(thick means the basin of A_β is a neighborhood of A_β)

By our paper in ETDS 2012, Q is equivalent to: for $\beta \notin [-2i, 2i]$
For what β are the Julia sets $J_\beta = J(S_\beta)$ and $J'_\beta = J(S_\beta^{-1})$ disjoint?
For such β, $A_\beta = J'_\beta$ is a thick attractor for S_β (and J_β is a thick attractor for S_β^{-1}).

A_β exists for $|\beta| > 2$ since for such β

$$|z| \geq 1 \Rightarrow |\pm\beta + \frac{1}{z}| \geq k > 1, \text{ where } k = |\beta| - 1.$$

Moreover $\beta + D$, $-\beta + D$ are disjoint (where D is the unit disc) so A_β has a trivial symbolic dynamics: $A_\beta \cong \{0,1\}^{\mathbb{N}}$ where f, g act respectively by i_0, i_1 where

$$i_0(z_0, z_1, \ldots) = (0, z_0, z_1, \ldots), \quad i_1(z_0, z_1, \ldots) = (1, z_0, z_1, \ldots).$$

Using the theory of Riley groups we'll extend this result to an open annular region $\mathcal{S} \subset \mathbb{C}$ (the Koebe slice) and explore the remaining cases $\beta \in \mathcal{K} \underset{\text{def}}{=} \partial \mathcal{S}$ and $\beta \in \mathcal{E} \underset{\text{def}}{=} \mathbb{C}^* - \overline{\mathcal{S}}$.

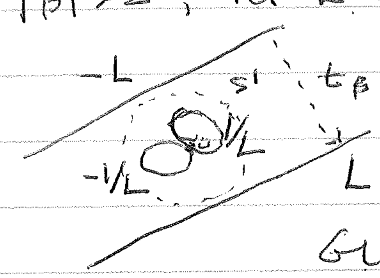
A Riley group is a Moebius group generated by 2 parabolic elements $z+a$ (fixing ∞) and $z/(bz+1)$ (fixing 0), $a \neq 0 \neq b$, especially the group

$$P_\beta = \langle t_\beta, u_\beta \rangle, \quad t_\beta(z) = z+\beta,\ u_\beta(z) = z/(\beta z+1).$$

P_β is closed related to the group G_β generated by S_β, namely the even-length words in $t_\beta^{\pm 1}, u_\beta^{\pm 1}$ are the even-length words in $f_\beta^{\pm 1}, g_\beta^{\pm 1}$ (for instance $f_\beta^2 = t_\beta u_\beta$, $f_\beta g_\beta = t_\beta u_\beta^{-1}$), so P_β, G_β are commensurable. As P_β is nonelementary, these groups have the same ordinary set Ω_β (defined as the largest invariant open set where P_β acts discretely).

When Ω_β is nonempty we get an orbifold $\Sigma_\beta = \Omega_\beta / P_\beta$.

EX $|\beta| > 2$, let L be the line $\mathrm{Re}(z/\beta) = \frac{1}{2}$. The circles $\pm 1/L$ and the lines $\pm L$ bound a fundamental domain for Σ_β, which contains the unit circle S^1.



Gluing shows Σ_β has genus 0 and 4 punctures.

The slice $\mathcal{S}$ is defined by: $\beta \in \mathcal{S}$ whenever t_β, u_β act freely on Ω_β so that Σ_β has genus 0 and 4 punctures.

One knows that $\mathcal{S}$ is open in $\mathbb{C}^*$ and homeomorphic to $S^1 \times \mathbb{R}$. Moreover $\partial \mathcal{S} = \mathcal{E}$ is a Jordan curve (Ohshika – Miyachi, 2008). We have a partial answer to Q.

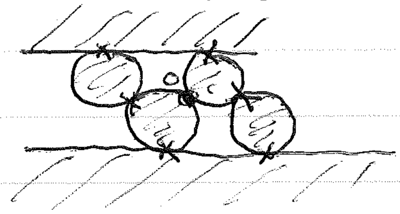
Theorem A_β exists for $\beta \in \mathcal{S}$ and does not exist for $\beta \in \mathcal{E}$.

Sketch of Pf For $|\beta| > 2$ define E_β as the geodesic in Ω_β (for the Poincaré metric) isotopic to S^1. For $\beta \in \mathcal{S}$ define the geodesic $E = E_\beta \subset \Omega_\beta$ by continuation from $|\beta| > 2$. One checks that E is a simple closed geodesic separating 0 from ∞, $\{\pm 1, \pm i\} \subset E = -E = 1/E$, and the $p(E)$, $p \in P_\beta$, are disjoint.

Let $N = N_\beta$ be the closed neighborhood of ∞ bounded by E_β. E is disjoint from $E \pm \beta$ so $f_\beta(N), g_\beta(N) = \beta + 1/N, -\beta + 1/N \subset \mathrm{Int}(N)$. Thus N is an S_β-block (as defined in our paper, op. cit.) so A_β exists. (Furthermore $\pm\beta + 1/N$ are disjoint so S_β has trivial symbolic dynamics.)

Note: another proof uses that the groups P_β, $\beta \in \mathcal{S}$, are quasiconformally conjugate, which implies that all the $\Lambda_\beta \overset{\text{def}}{=} \Lambda(P_\beta) = \Lambda(G_\beta)$ are Cantor sets. Likewise the S_β and $S'_\beta = A_\beta$ are Cantor sets.

The case $\beta \in \mathcal{E}$ uses certain cusps $c \in \mathcal{E}$ for which P_c has an additional parabolic (that is a parabolic element not conjugate to one of the parabolic generators). Such a cusp c is the endpoint of a pleating ray in $\mathcal{S}$, corresponding to a particular class of simple closed geodesics in Σ_β that degenerates as $\beta \to c$ along the ray (in the sense that the length of the geodesic approaches zero). This geodesic is characterized by a rational number modulo 2 $p/q \in \mathbb{Q}/2\mathbb{Z}$, where the geodesic cuts E_β $q > 0$ times and winds p times (in an appropriate sense). Pleating rays were studied by Keen and Series* and the corresponding cusp groups P_c were studied by Wright.** Wright finds $2q$ ~~conformal~~ nonoverlapping discs in $\hat{\mathbb{C}}$ that are cyclically arranged and tangent in successive pairs, with 2 additional tangencies at 0 and at ∞. (For $p/q = 1/3$ this configuration is



where the $2q = 6$ x's mark the successive tangencies.) For β near c on the pleating ray there are $2q$ slightly overlapping discs whose union contains E_β. With this in mind, we define E_c to be the union of the $2q$ circular arcs joining the successive x's, so E_c lies in the union of the $2q$ discs. Then $E_c = -E_c = 1/E_c$ and we define N_c as before. One checks that $J_c' \subset N_c$ but $J_c' \cap E_c$ is the finite set of x's. This set is also $J_c \cap E_c$ so $|J_c' \cap J_c| = 2q$. Hence P_c has no strong attractor. As the existence of a strong attractor is a stable property and as the cusps c are known to be dense in $\mathcal{E}$, we see that A_β does not exist for $\beta \in \mathcal{E}$.

*Proc London Math Soc, 1994 **in Spaces of Kleinian Groups, London Math Soc Lecture Notes vol 329, 2005.

This theorem exploits the fact that the semigroups S_β, $\beta \in \mathcal{S}$, contains representatives for the simple closed geodesics of interest in Σ_β.

While all the P_β, $\beta \in \mathcal{S}$, are alike there is a dichotomy for the P_β, $\beta \in \mathcal{C}$. All these P_β are discrete groups and free on the generators t_β, u_β but their limit sets vary drastically. For $\beta = c$ a cusp, Λ_c is a gasket (that is the ordinary set Ω_c is a union of disjoint discs, including the $2q$ discs considered by Wright) and P_c is geometrically finite. For β not a cusp, however, $\Lambda_\beta = \hat{\mathbb{C}}$ (so Ω_β is empty) and P_β is geometrically infinite. One expects a similar dichotomy for the semigroups S_β, namely that $J_\beta \cap J'_\beta$ is infinite when $\beta \in \mathcal{C}$ is not a cusp.

The P_β for $\beta \in \mathcal{E}$ vary wildly. Only countably many are discrete but these include many interesting examples associated to 2-bridge link complements. As far as we know, $J_\beta = \Lambda_\beta = J'_\beta$ for $\beta \in \mathcal{E}$. This is known when $\boxed{id \in \overline{S_\beta}}$ (especially when S_β contains a torsion element). This condition on β may be dense (in $\mathcal{E}$), in which case there are no A_β, $\beta \in \mathcal{E}$.