Pattern-forming fronts in a Swift-Hohenberg equation with directional quenching

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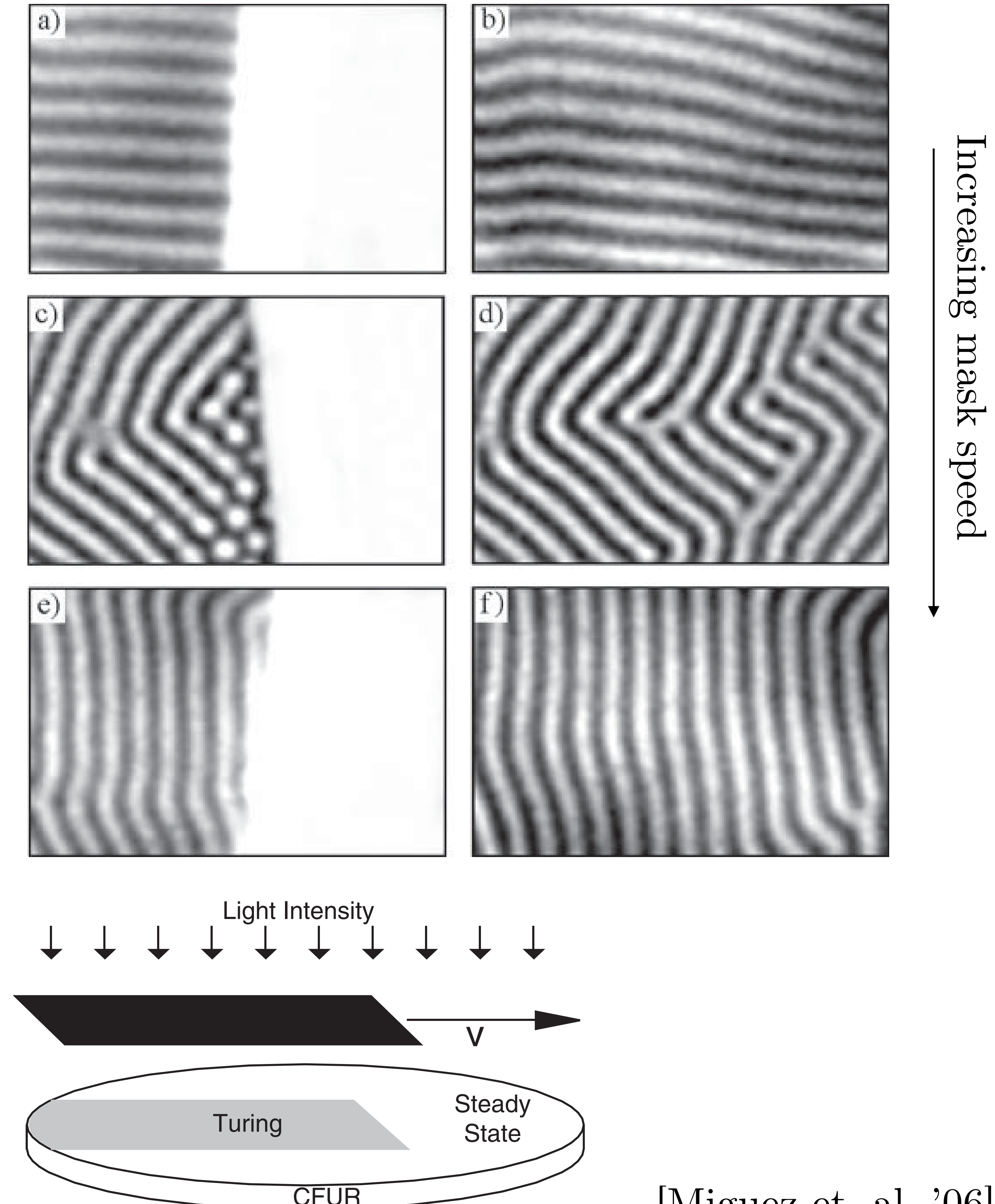
A motivating example: Light Sensitive RD-system

- ClO_2-I-Malonic Acid (CDIMA)- gel on top of well-stirred mixture
- Light suppresses Turing instability
- Mask speed *selects* pattern and *mediates* defects
- Experimental model for growing domains
- Modeled by a moving reaction-diffusion system



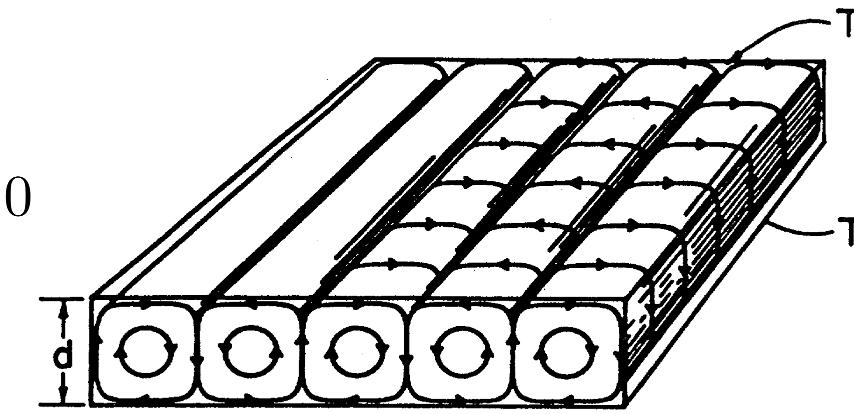
[Miguez et. al. '06]

FIG. 1. Schematic of the experiment. A moving opaque mask image creates a growing shadow domain where Turing patterns can develop. In the illuminated domain the pattern is suppressed.

We'll consider the Swift-Hohenberg equation

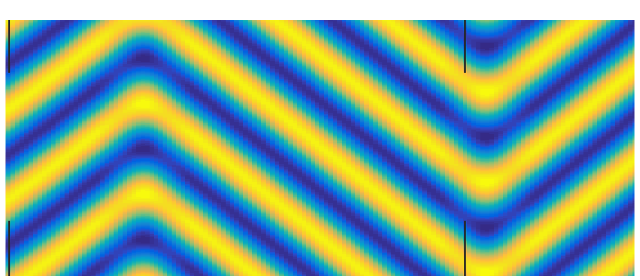
[SH-'77], [Cross, Hohenberg '93]

$$u_t = -(1+\Delta)^2 u + \mu_0 u - u^3, \quad u : \mathbb{R}^n \to \mathbb{R},$$

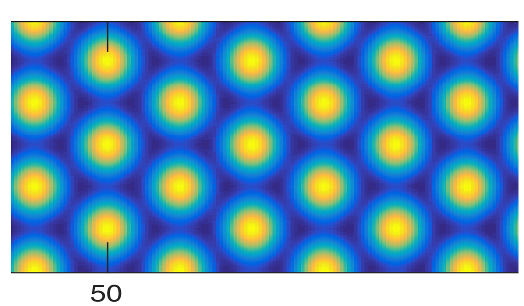


- u - order parameter, measures state of system
- μ_0 -bifurcation/"onset parameter: $u \equiv 0$ stable/unstable for $\mu_0 \lessgtr 0$
- First developed for Rayleigh-Benard convection
- Subsequently used as a "normal form" model many phenomena:
  - other fluid systems, plant phyllotaxis, liquid crystals, crystallization, etc...
- Some similar behavior to reaction-diffusion systems
- Nice starting point because much is rigorously known:
  - Existence/stability of homogeneous patterns:
  - Fronts, slow-dynamics, localized patterns

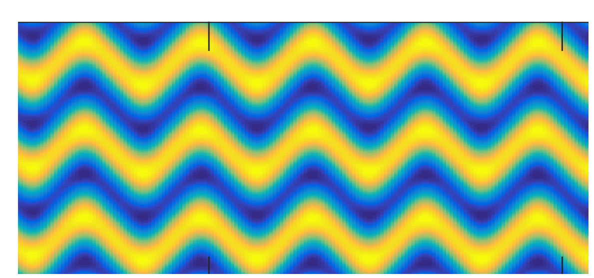
Grain Boundaries



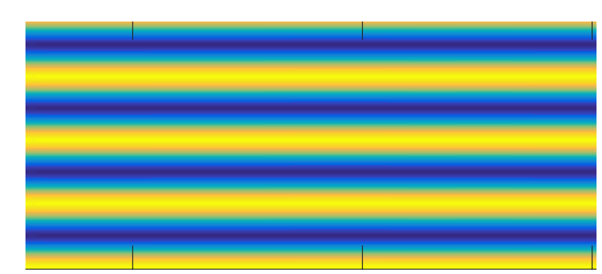
Hexagons



Zig-Zags



Stripes



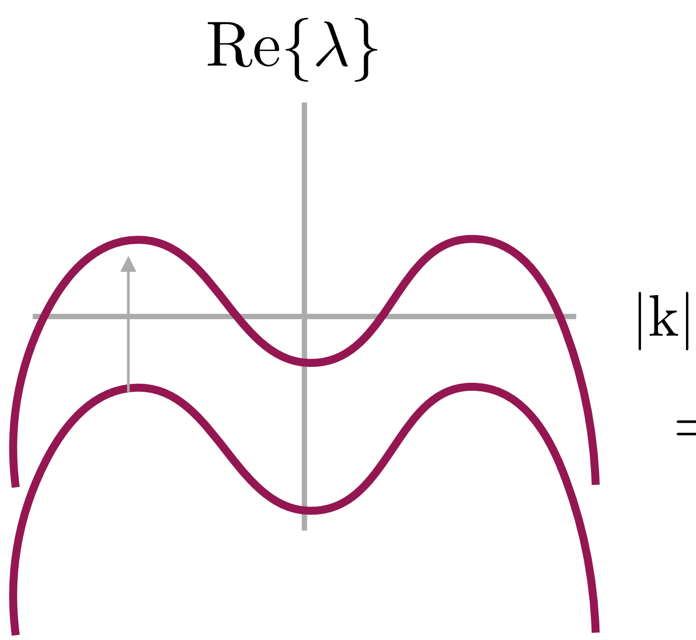
Swift-Hohenberg equation

$$u_t = -(1+\Delta)^2 u + \mu_0 u - u^3, \quad u : \mathbb{R}^n \to \mathbb{R},$$

- Much known about system at onset $0 < \mu_0 \ll 1$

Turing instability: insert $u = r e^{ik\cdot x + \lambda t}$ into linear equation yields $\lambda = -(1-|k|^2)^2 + \mu_0$

$\mu_0 \nearrow 0$



$\Longrightarrow$

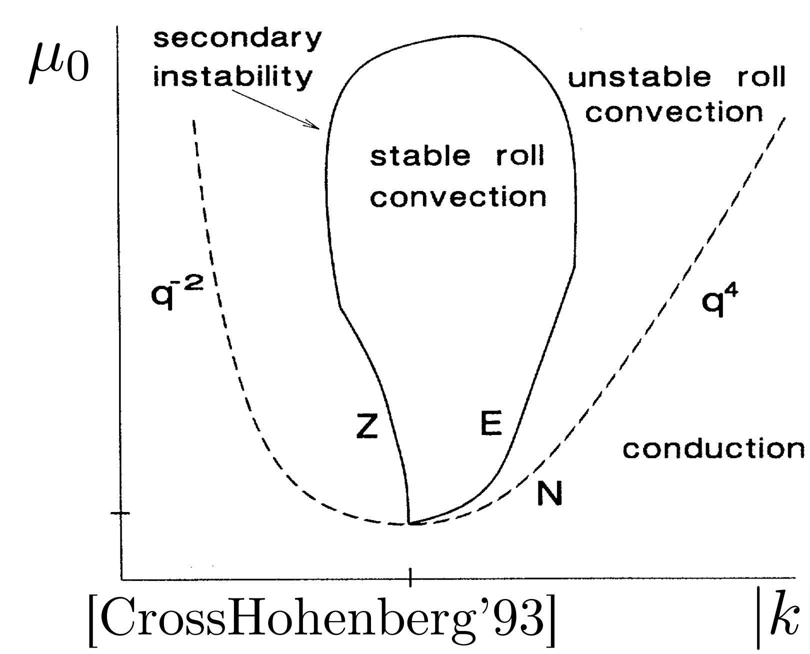
Bifurcation of family of "roll"/stripe equilibrium states in nonlinear equation

$$u_p(x_1) = \sqrt{4(\mu_0 - \kappa)/3}\cos(|k|x_1) + \mathcal{O}(|\mu_0 - \kappa^2|^{3/2}),$$

$$\kappa = |k|^2 - 1, \ |k| \sim 1$$

Rotational invariance -> all orientations of stripes are solutions

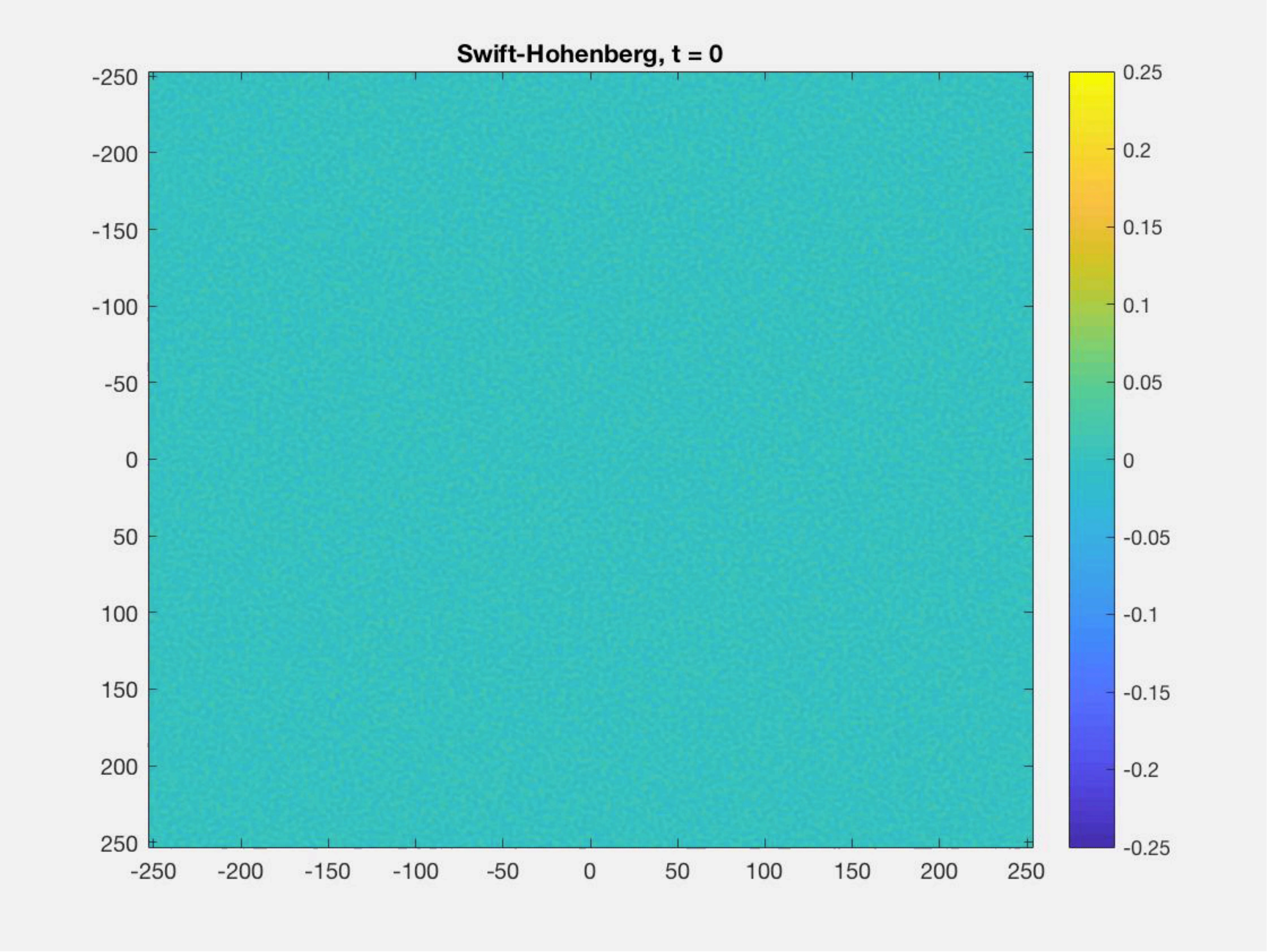
$$u_p(k \cdot x; k)$$



[CrossHohenberg'93]

Swift-Hohenberg equation

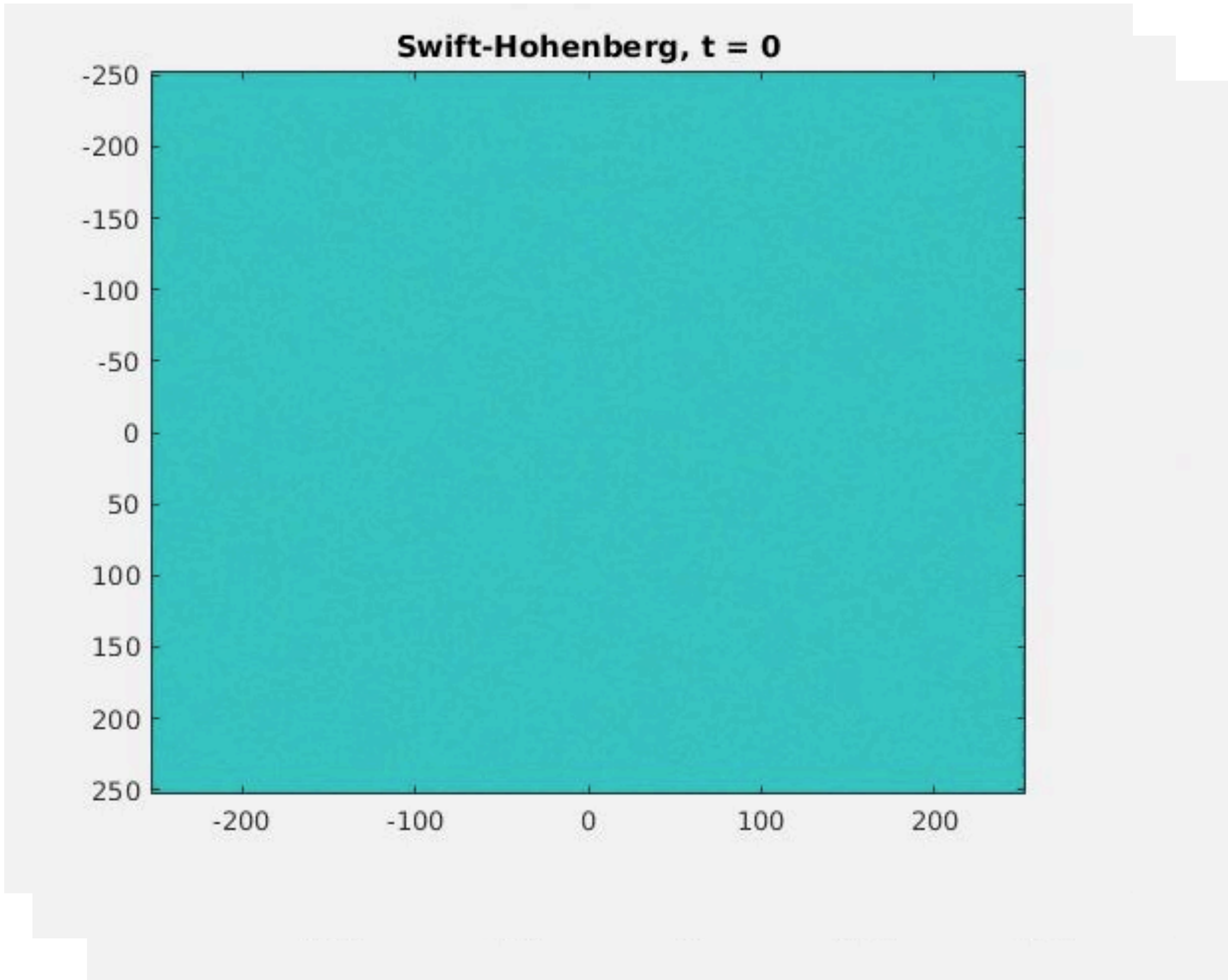
$$u_t = -(1+\Delta)^2 u + \mu_0 u - u^3, \quad u : \mathbb{R}^2 \to \mathbb{R},$$



Quenched Swift-Hohenberg equation

$$u_t = -(1+\Delta)^2 u + \mu(x-ct)u - u^3, \quad \mu(\xi) = -\mu_0 \mathrm{sgn}(\xi)$$

- Inhomogeneity changes stability of trivial state for $x - ct \gtrless 0$

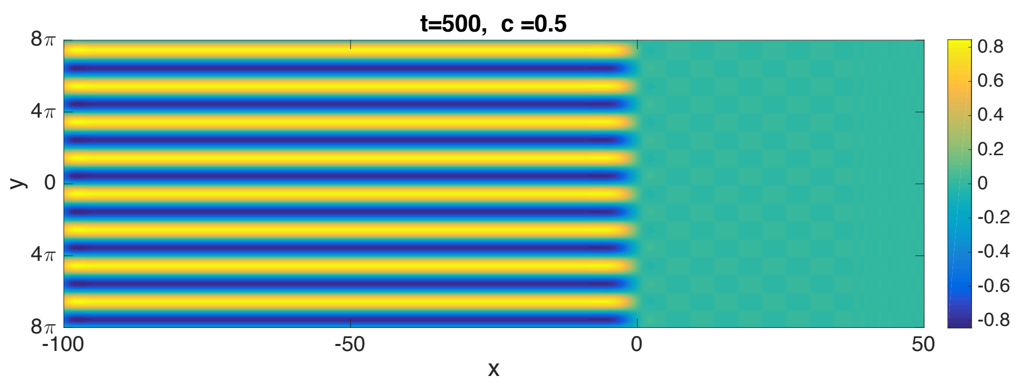


- *Main question: How does the in-homogeneity control, or select, patterns?*
- Similar behavior to experimental RD system

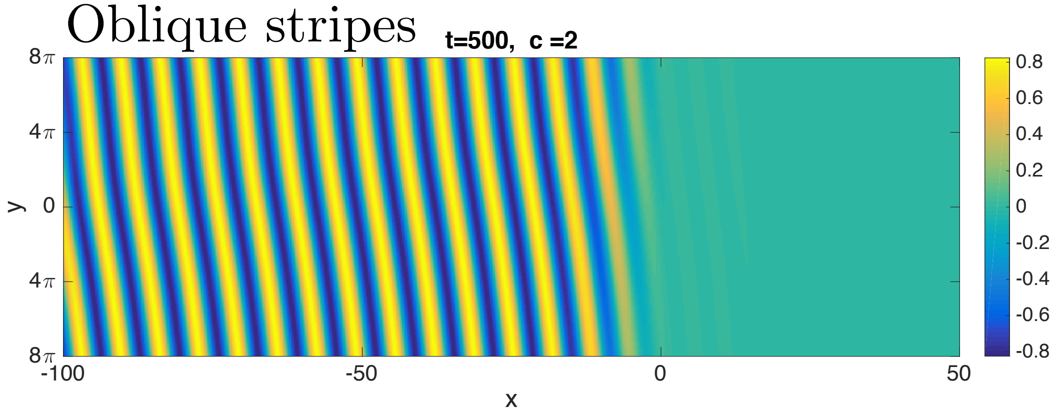
Swift-Hohenberg

Light Sensing RD system

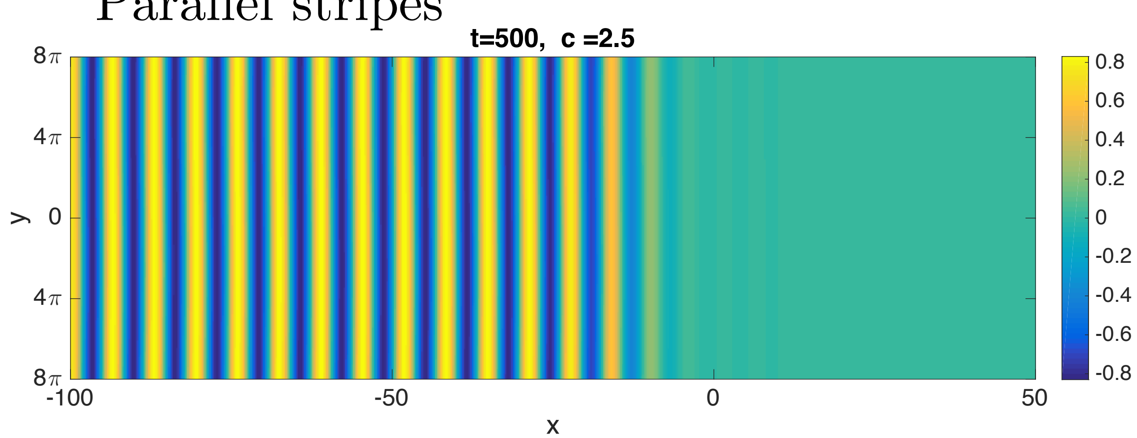
Perpendicular stripes



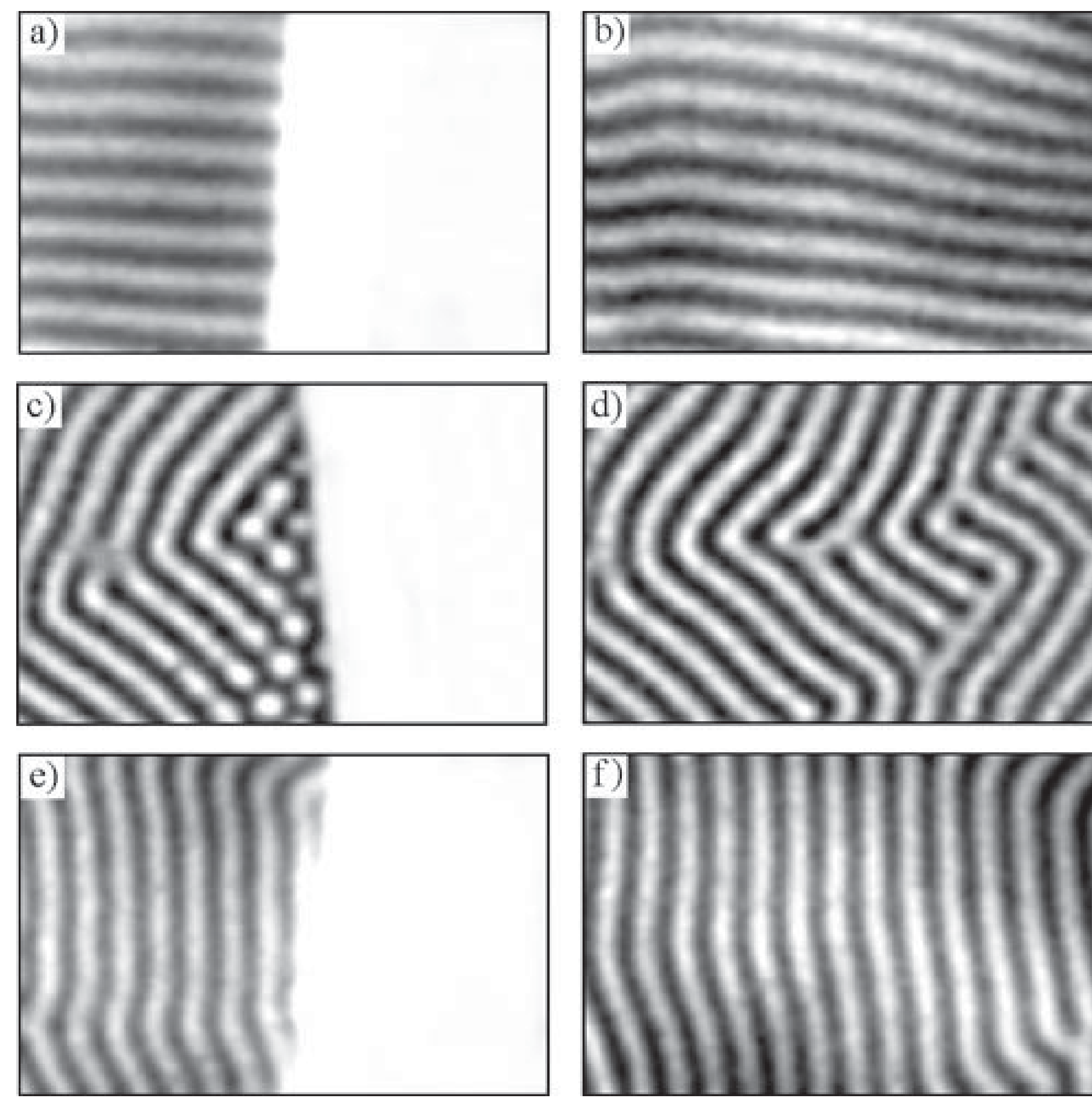
Oblique stripes



Parallel stripes

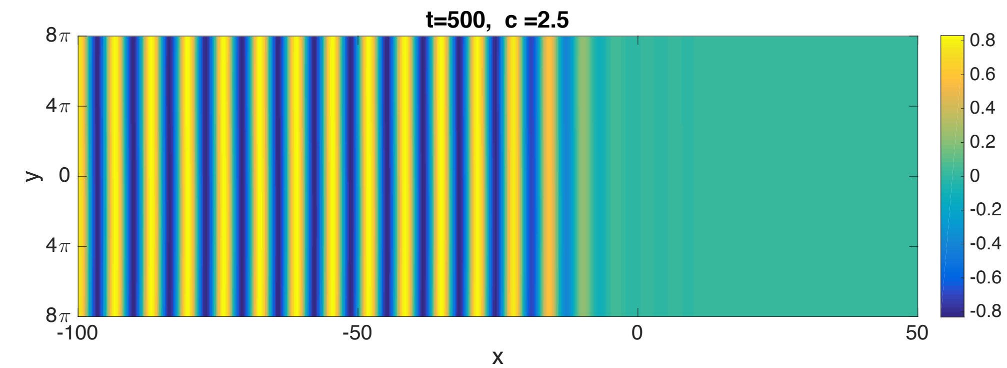


Increasing quenching c



Outline

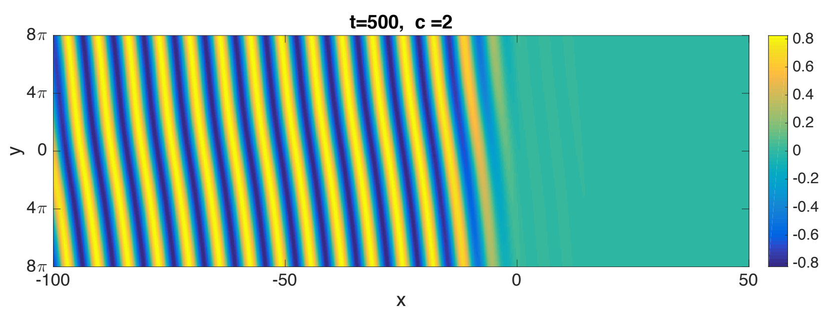
- 1-D patterns

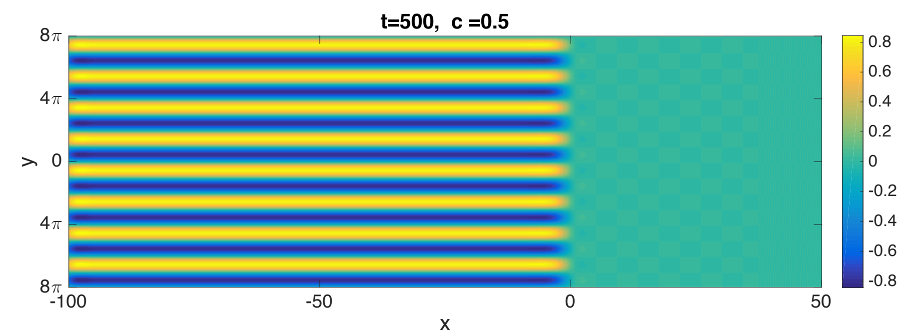


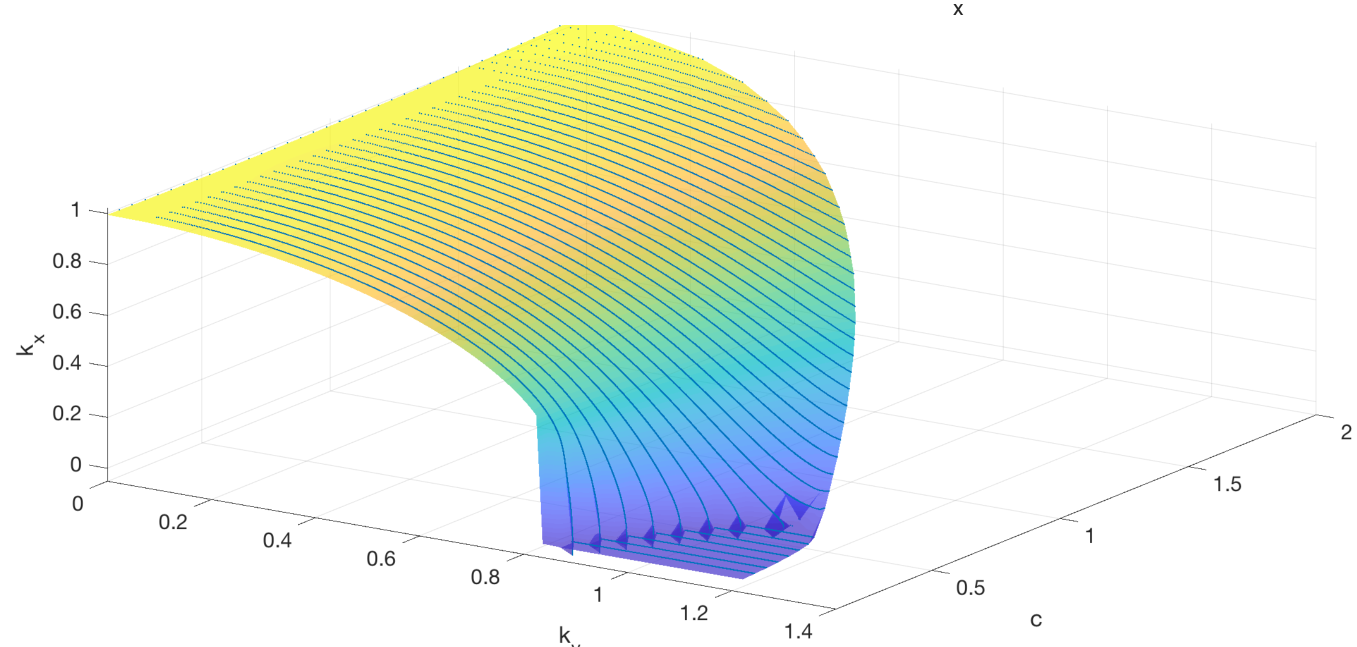
  - Spatial dynamics:

    Center manifold, invariant foliations, heteroclinic bifurcations, and Melnikov integrals

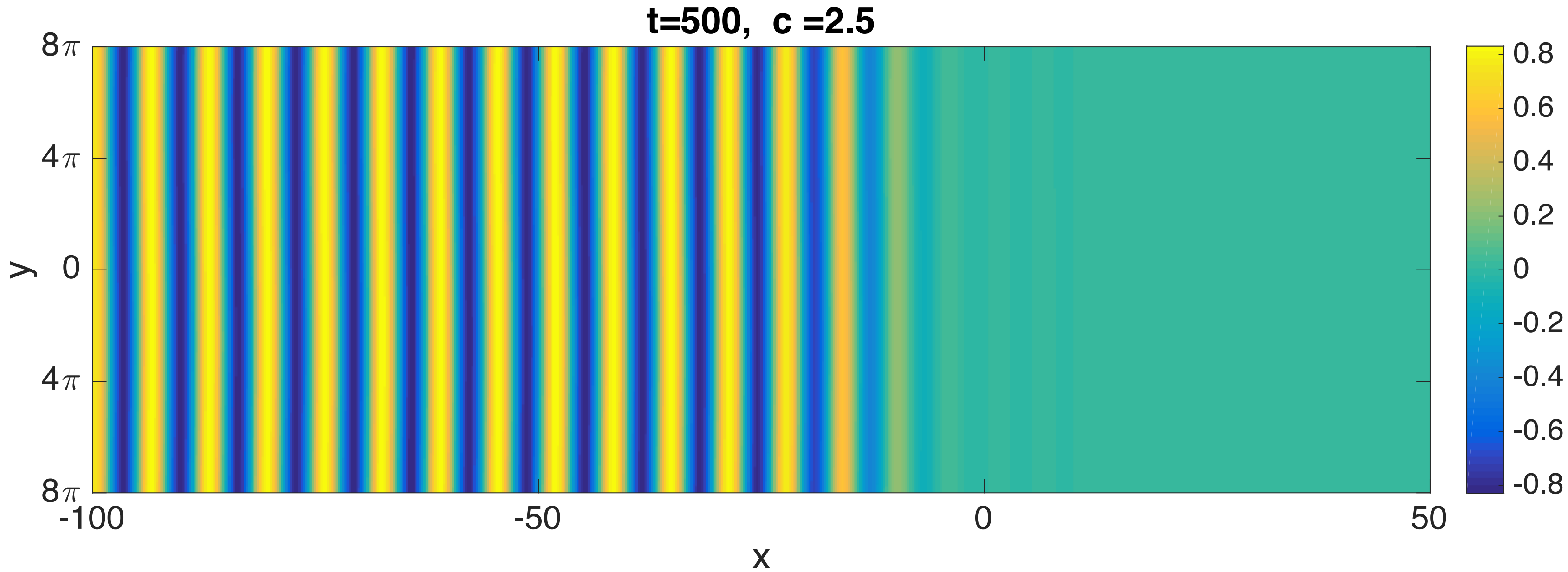
- 2-D patterns and beyond:





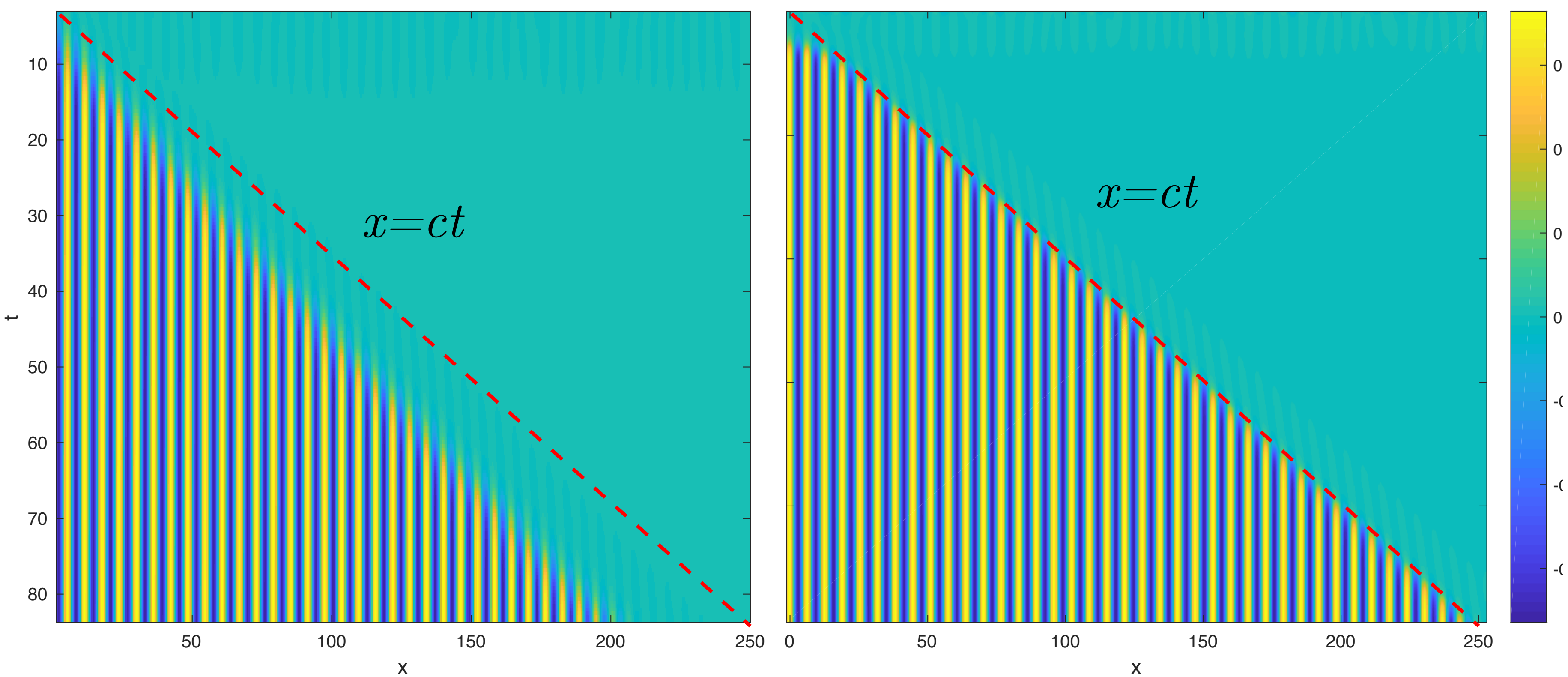


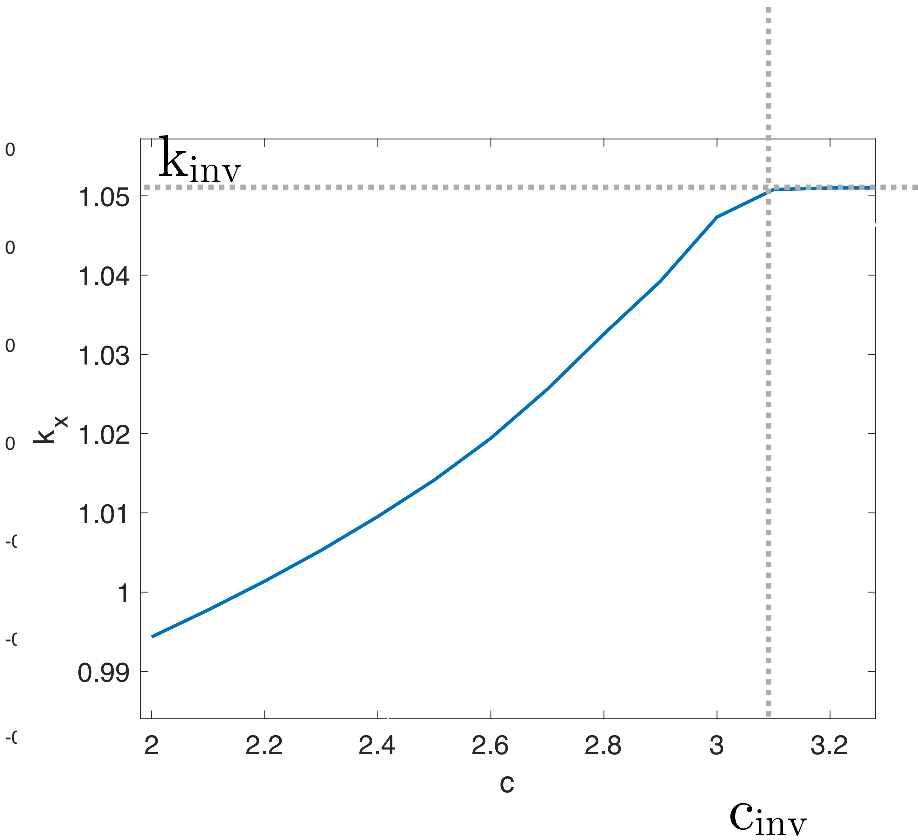
In this talk, we'll focus on1-D patterns:



1-D patterns in Swift-Hohenberg

$$u_t = -(1+\partial_x^2)^2 u + \mu(x-ct)u - u^3 \qquad \mu(\xi) = \mu_0 \mathrm{sgn}(-\xi)$$





$c > c_{\mathrm{inv}}$: pattern selected by unstable homogeneous state behind inhomogeneity.

$c < c_{\mathrm{inv}}$: pattern wants to invade faster than you're letting it

Interface has no effect on pattern for $c > c_{\mathrm{inv}}$

Curves k(c) give mechanism and prescription for control in fabrication of materials

Fast speeds:

[RG, Scheel; to appear]

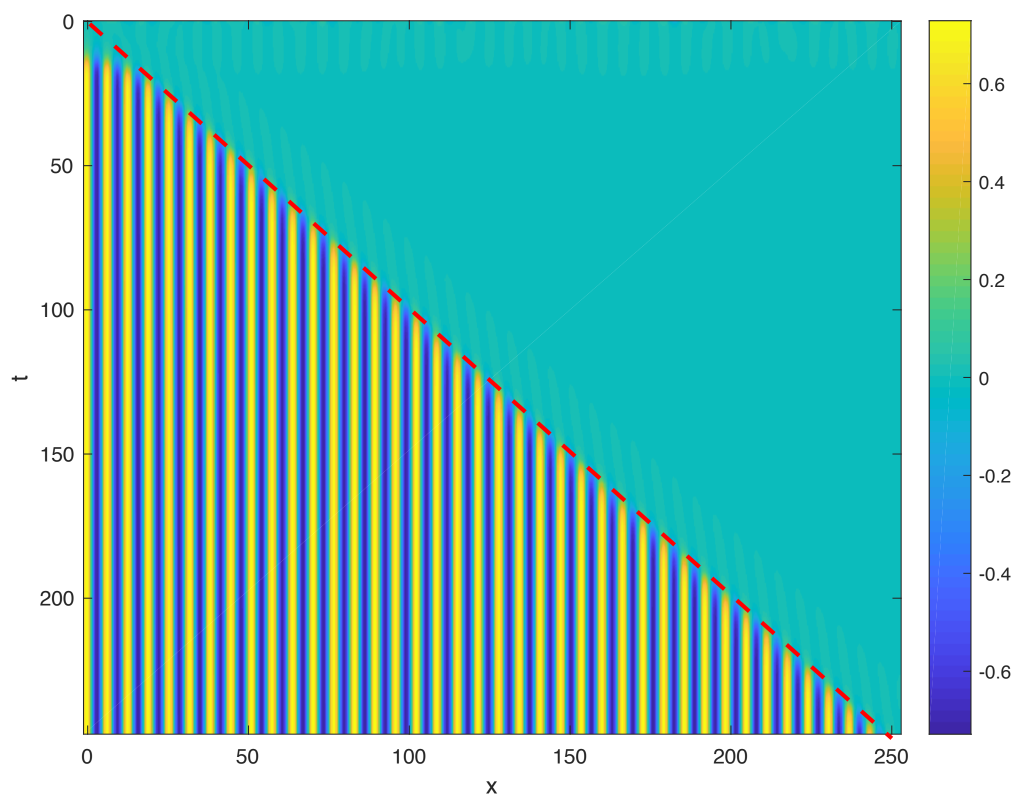
$$u_t = -(1+\partial_x^2)^2 u + \mu(x-ct)u - u^3$$

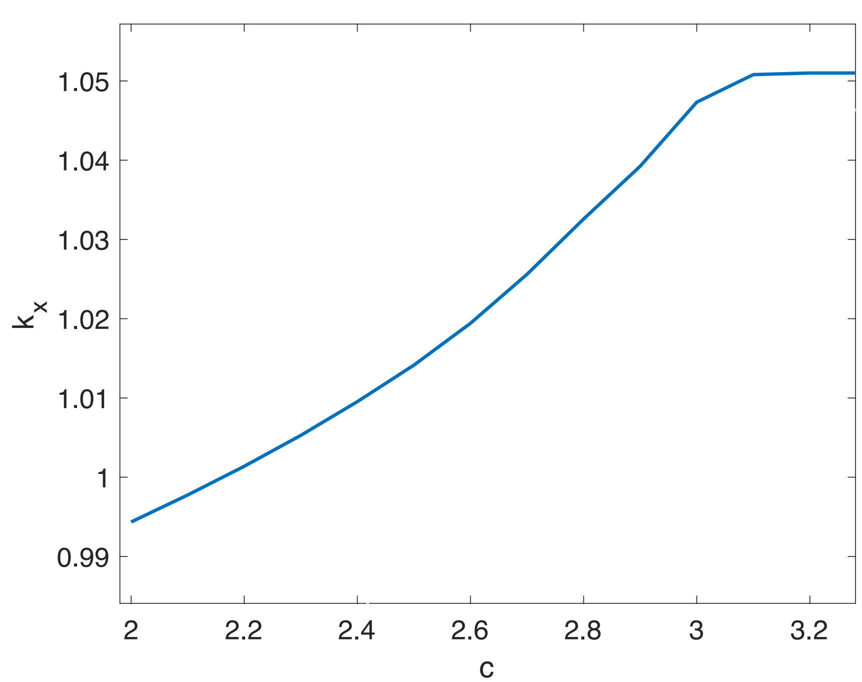
Study for speeds near detachment point $c \sim c_{\text{inv}} = 4\sqrt{\mu_0}$

Look for small amplitude solutions with onset multiple scaling: $\mu_0 = \epsilon^2, c = \epsilon\tilde{c}, 0 < \epsilon \ll 1$

Thm: For ϵ and $4-\tilde{c} > 0$ *sufficiently small, there exists a pattern forming front with wavenumber*

$$k = 1 + \epsilon\tilde{\gamma}, \quad \tilde{\gamma} = \tilde{\gamma}(4-\tilde{c}, \epsilon)$$





Our approach: spatial dynamics

- Study pattern-forming front solutions as heteroclinic orbits of a non-autonomous dynamical system with spatial variable, $\xi = x - ct$ as evolution variable.
    - Connect roll solution at $\xi = -\infty$ with trivial solution at $\xi = +\infty$
- (center-manifold reduction): Use onset scaling to look for small, bounded solutions, -> use center-manifold techniques to get leading order dynamics.
- (Shooting): Overlay phase spaces $\xi \gtrless 0$ to find intersection of relevant invariant manifolds
- (persistence/transverse unfolding): Invariant foliations to lift leading order dynamics to full infinite-dimensional system

But first: the homogeneous equation

- "*Propagating fronts and the center-manifold theorem*" [Eckmann, Wayne, '91]*:

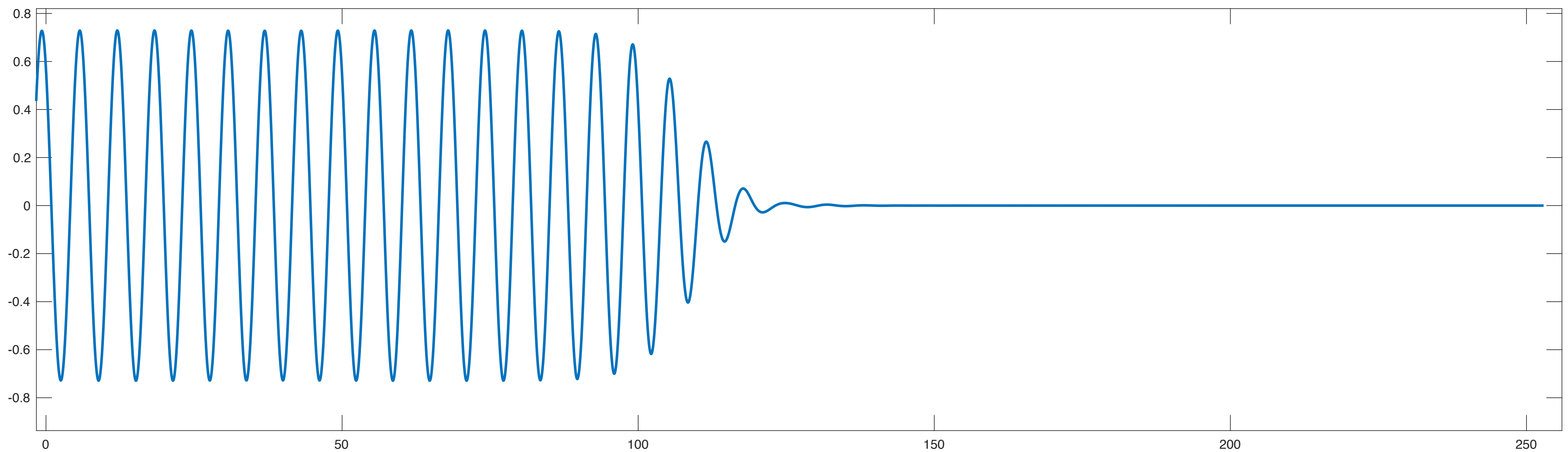
$$u_t = -(1+\partial_x^2)^2 u + \epsilon^2 \tilde{\mu}_0 u - u^3 \qquad \text{Roll solutions:} \qquad u_p(kx;k)$$

- Look for solutions of the form:

$$u(x,t) = W(x, x-ct), \text{ with } W(x,\xi) = W(x+2\pi/k, \xi) \qquad c = \epsilon\tilde{c} \qquad k = 1+\epsilon\tilde{\gamma}$$

$$\lim_{\xi\to\infty} W(x,\xi) = 0, \qquad \lim_{\xi\to-\infty} W(x,\xi) = u_p(kx;k)$$

Theorem: For $\tilde{c} > 4\tilde{\mu}_0 > 0$, *and* $0 < \epsilon \ll 1$, *and a family of wavenumbers* k *in a neighborhood of 1, there exists a traveling front solution, connecting a roll* $u_p(kx;k)$ *to the homogeneous (unstable!) equilibria* $u = 0$.



*See also [Haragus,Schneider '99, Doelman et. al 2003, Faye,Holzer 2015]

Homogeneous equation

- Fourier transform: $W(x,\xi) = \sum_{n\in\mathbb{Z}} W_n(\xi)\mathrm{e}^{-\mathrm{i}nkx}, \quad u_p(kx) = \sum_n S_n \mathrm{e}^{-\mathrm{i}nkx}$
- Coupled system of ODEs:

$$[-(1+(\mathrm{i}kn+\partial_\xi)^2)^2 + \epsilon\tilde{c}\partial_\xi + \epsilon^2\tilde{\mu}_0]W_n(\xi) = \sum_{p+q+r=n} W_p(\xi)W_q(\xi)W_r(\xi)$$

- Write these higher order equations as first order systems in phase space $\mathcal{E}_0 = \oplus_{n=0}^{\infty}\mathbb{C}^4$

$$\partial_\xi X_n = M_n X_n + F_n(X), \qquad X = (X_n)_{n\in\mathbb{Z}}, \quad X_n = (W_n, \partial_\xi W_n, \partial_\xi^2 W_n, \partial_\xi^3 W_n)^T$$

$$M_n = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ A & B & C & D \end{pmatrix}, \quad F_n(X) = (0,0,0,\sum_{p+q+r=n} X_{p,0}X_{q,0}X_{r,0})^T,$$

$$A = -(1-(kn)^2)^2 + \epsilon^2\tilde{\mu}_0, \quad B = 4\mathrm{i}kn(1-(kn)^2) + c, \quad C = 6(kn)^2 - 2, \quad D = 4\mathrm{i}kn.$$

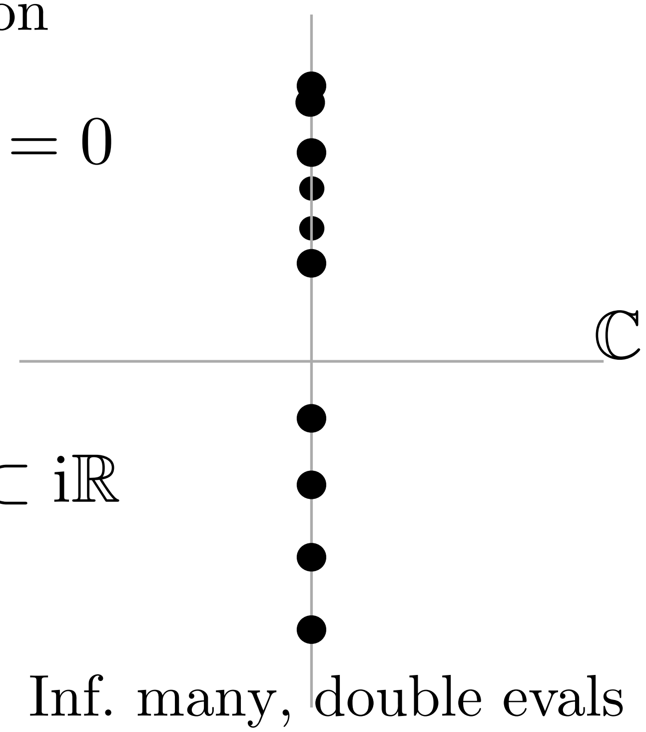
- Look for bounded solutions (in ξ), near origin
- Hence need to study spectrum of each M_n for $\epsilon \sim 0$

—> Look for heteroclinic orbits between equilibria

Spectral information

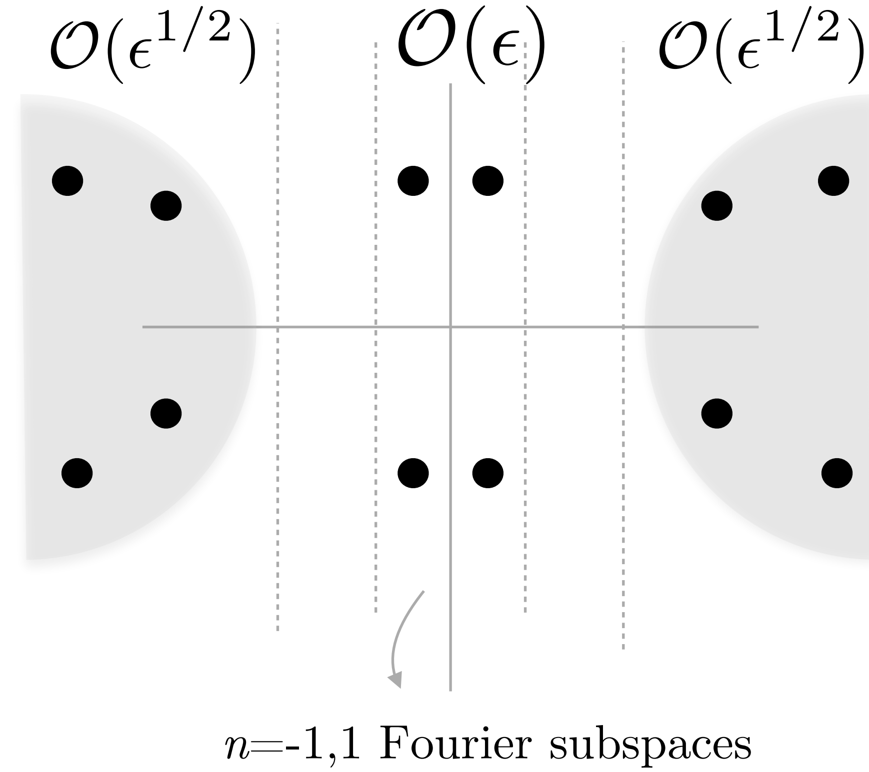
Spectrum of linearization about origin:

$\epsilon = 0$



$\mathrm{spec}(M) \subset \mathrm{i}\mathbb{R}$

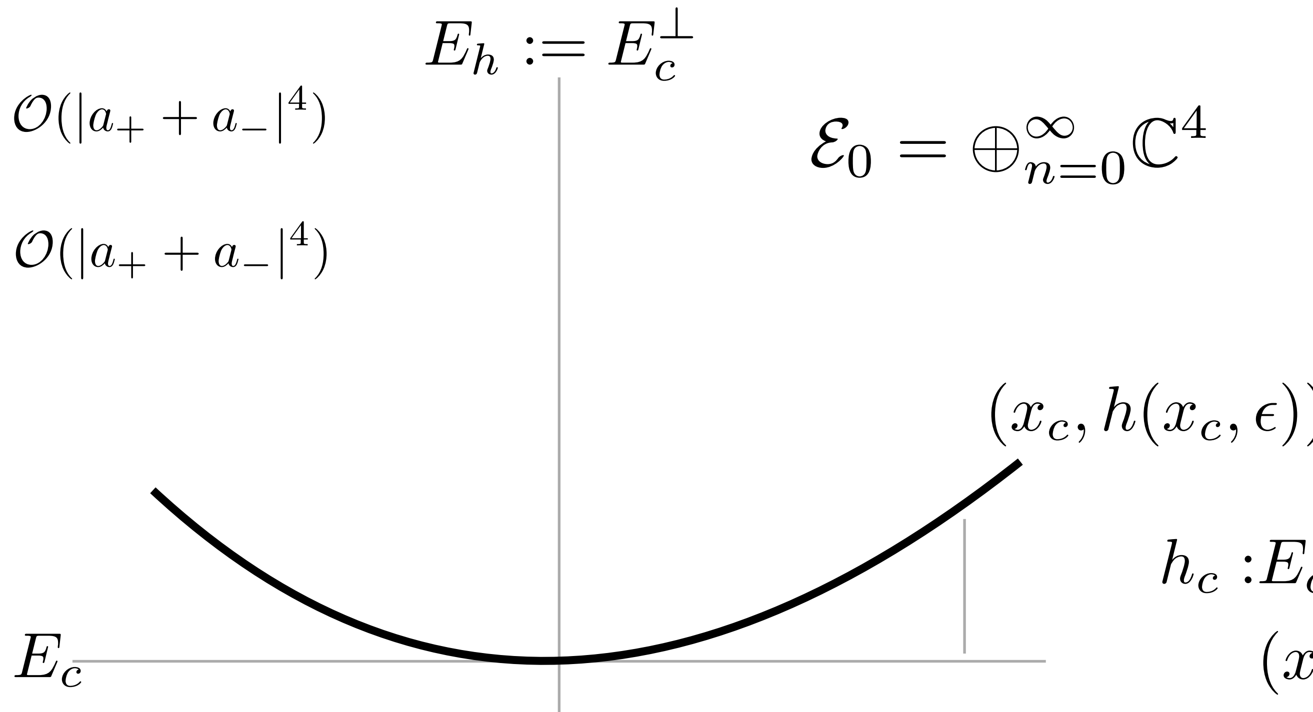
$0 < \epsilon \ll 1$



Center manifold theorem [EW,'91]: After symmetry reduction, there exists a two-dimensional invariant manifold, tangent to the $\mathcal{O}(\epsilon)$*-eigenspace, with leading order dynamics:*

$$\frac{da_+}{d\xi} = \nu_+ a_+ - 3c_+(a_+ + a_-)|a_+ + a_-|^2 + \mathcal{O}(|a_+ + a_-|^4)$$

$$\frac{da_-}{d\xi} = \nu_- a_- - 3c_-(a_+ + a_-)|a_+ + a_-|^2 + \mathcal{O}(|a_+ + a_-|^4)$$



$$\mathcal{E}_0 = \oplus_{n=0}^{\infty} \mathbb{C}^4$$

$$h_c : E_c \times \mathbb{R} \to E_h$$

$$(x_c, \epsilon) \mapsto h_c(x_c, \epsilon)$$

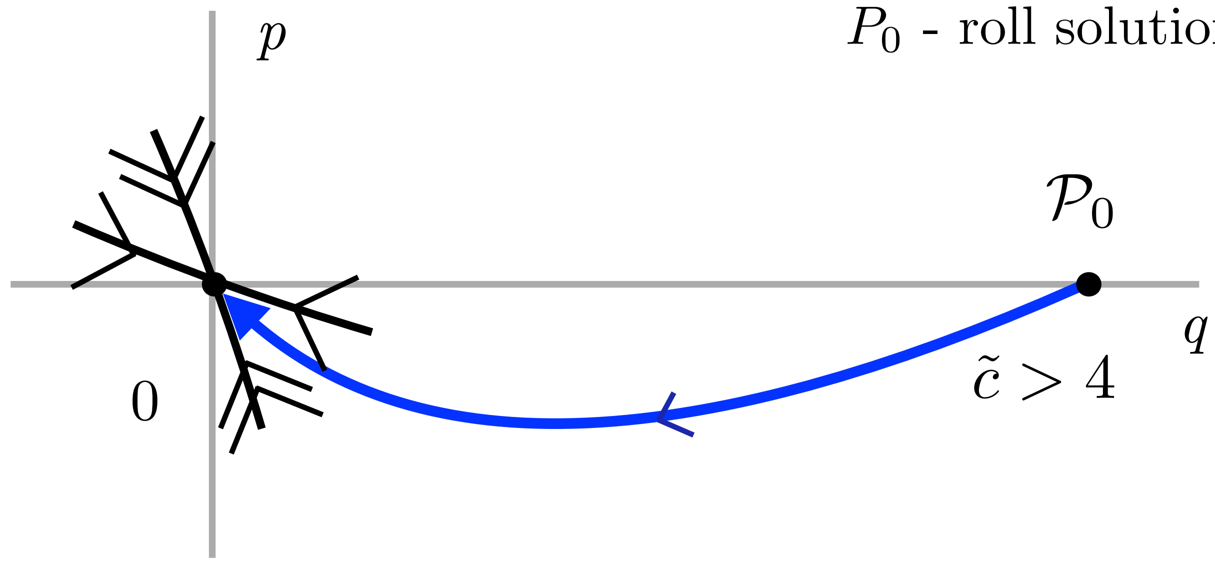
Dynamics on center manifold and fronts

- After a normal-form transformation $(x_+, x_-) \mapsto (p, q)$, and rescaling time $\zeta = \epsilon\xi$ we obtain the dynamics:

$$\frac{dq}{d\zeta} = p + \mathcal{O}(\epsilon), \quad (p, q) \in \mathbb{C}^2$$

$$\frac{dp}{d\zeta} = \frac{1}{4}\left(-\tilde{\mu}_0 q - \tilde{c}p + 3q|q|^2\right) + \mathcal{O}(\epsilon),$$

P_0 - roll solution



- Equilibrium $\mathcal{P}_0$ corresponds to a periodic orbit.
- Phase-invariance of the system under rotations $(q, p) \mapsto e^{i\theta}(q, p) \implies e^{i\theta}\mathcal{P}_0$ also an equilibrium
- Can conclude one parameter family of heteroclinics in leading order system $\epsilon = 0$
- Use normal hyperbolicity to show persistence for $\mathcal{O}(\epsilon)$ pertubations (i.e. full center-manifold dynamics)

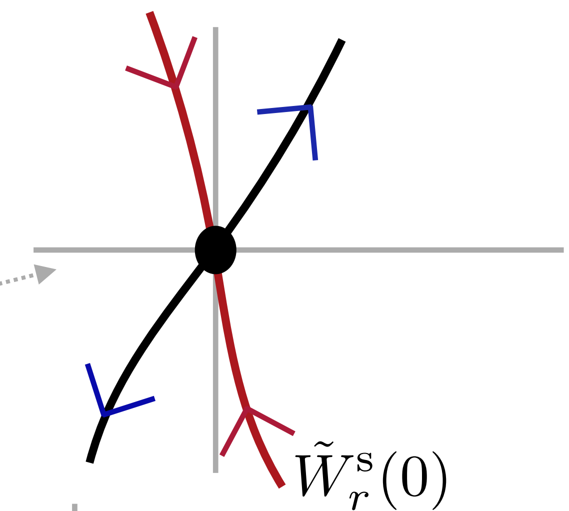
Backed to quenched system

$$u_t = -(1+\partial_x^2)^2 u + \epsilon^2 \tilde{\mu}_0 \mathrm{sgn}(-(x - \epsilon \tilde{c}\, t)) - u^3$$

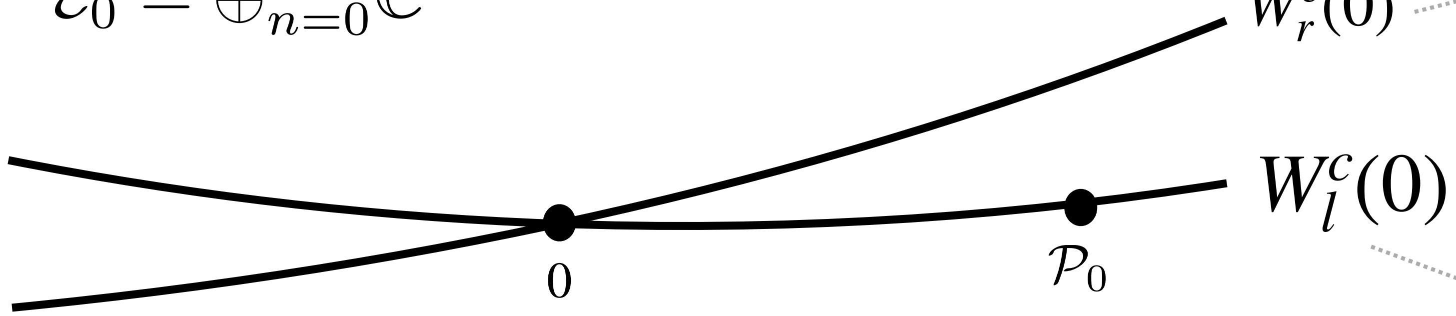
- Now ODE's are non-autonomous:

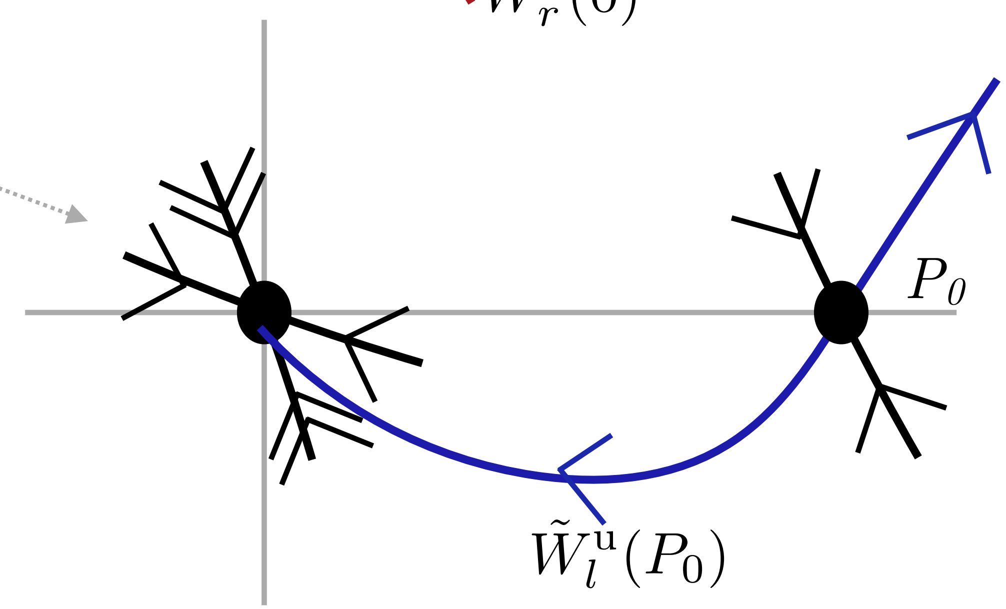
$$[-(1+(\mathrm{i}kn+\partial_\xi)^2)^2 + \epsilon\tilde{c}\partial_\xi + \epsilon^2\tilde{\mu}_0 \mathrm{sgn}(\xi)]W_n(\xi) = \sum_{p+q+r=n} W_p(\xi)W_q(\xi)W_r(\xi)$$

- Perform the same analysis above for both $\xi \gtrless 0$
    - Obtain two center manifolds $W_r^c(0), W_l^c(0)$



$$\mathcal{E}_0 = \oplus_{n=0}^{\infty} \mathbb{C}^4$$



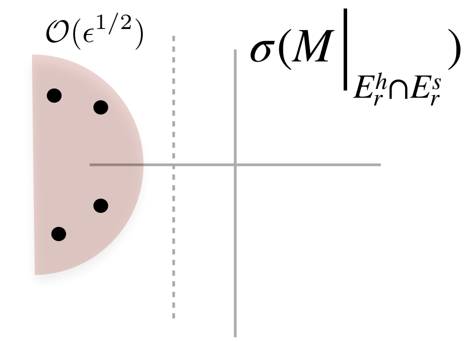


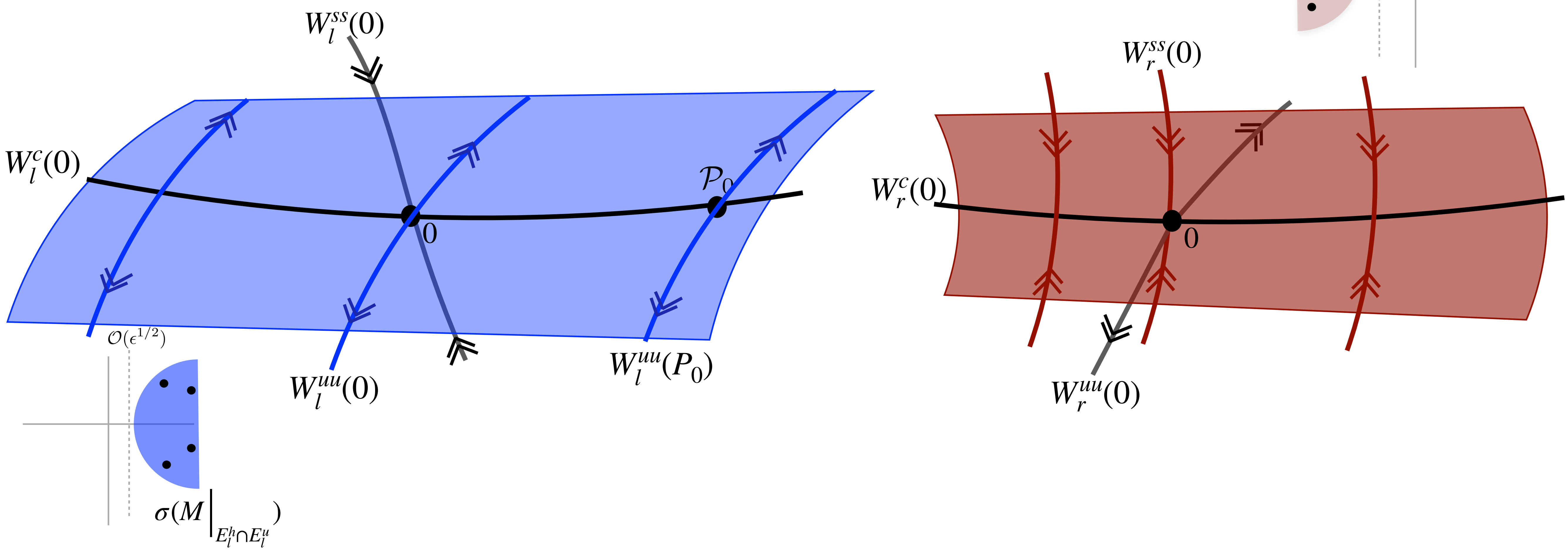
- Find orbit, in $\xi < 0$-dynamics, connecting $\mathcal{P}_0$ with stable manifold of origin in $\xi > 0$-dynamics

- Think of $\tilde{W}_r^{\mathrm{s}}(0)$ as boundary or target set to shoot $\tilde{W}_l^{\mathrm{u}}(P_0)$ at.

Invariant manifolds and foliations

- Center-manifolds only intersect trivially
    - Need to use hyperbolic dynamics normal to center manifolds





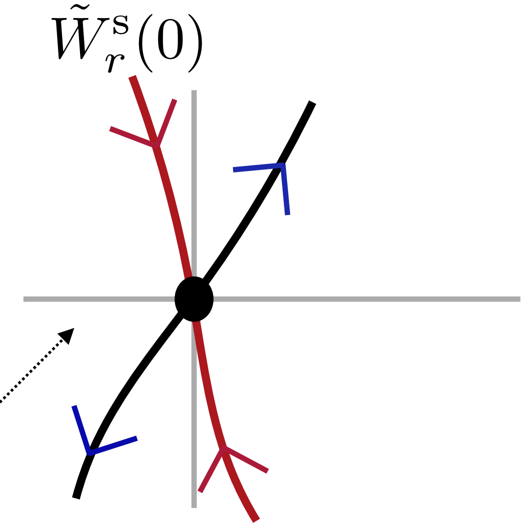
- Normal dynamics "foliated" by strong stable/unstable fibers, use these to find intersections

$$\mathcal{F}^{ss}_r = \bigcup_{w \in W^c_r(0)} \mathcal{F}^{ss}_{r,w}, \quad \mathcal{F}^{uu}_l = \bigcup_{w \in W^c_l(0)} \mathcal{F}^{uu}_{l,w} \qquad \Phi_\xi(\mathcal{F}^j_{r/l,w}) \subset \mathcal{F}^j_{r/l,\Phi_\xi(w)}$$

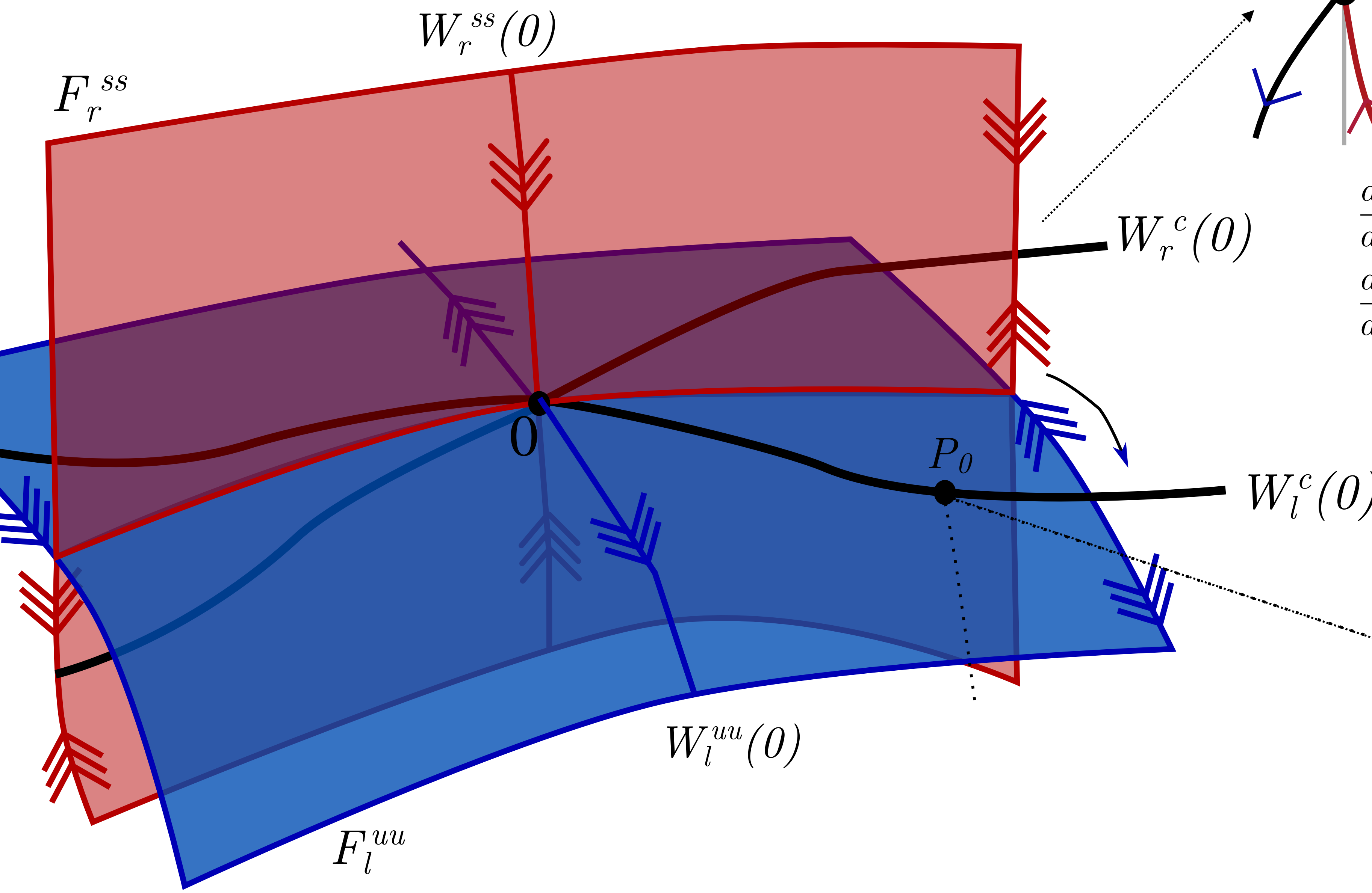
- Describe invariant manifolds by fibers

$$W^{uu}_l(\mathcal{P}_0)) = \mathcal{F}^{uu}_{l,\mathcal{P}_0}, \quad W^{ss}_r(0) = \mathcal{F}^{ss}_{r,0}, \quad W^s_r(0) = \bigcup_{w \in \tilde{W}^s_r(0)} \mathcal{F}^{ss}_{r,w}$$

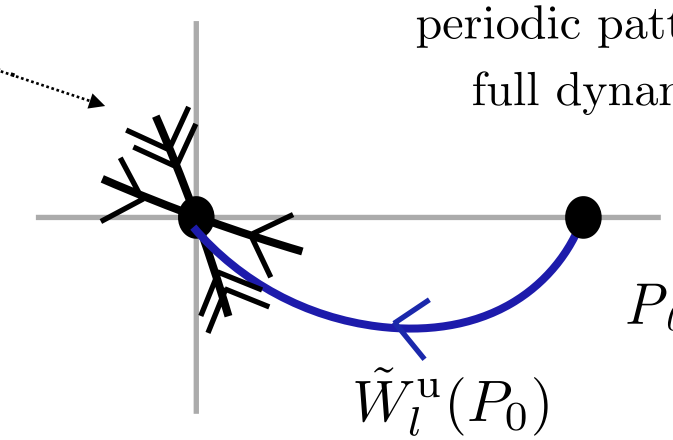
- Find orbit, in $\xi < 0$ -dynamics connecting $\mathcal{P}_0$ with stable manifold of origin in $\xi > 0$ -dynamics



$$\frac{dq}{d\zeta} = p + \mathcal{O}(\epsilon),$$
$$\frac{dp}{d\zeta} = \frac{1}{4}\left(q - \tilde{c}p + 3q|q|^2\right) + \tilde{\gamma}(2\mathrm{i}p + \tilde{\gamma}q) + \mathcal{O}(\epsilon)$$



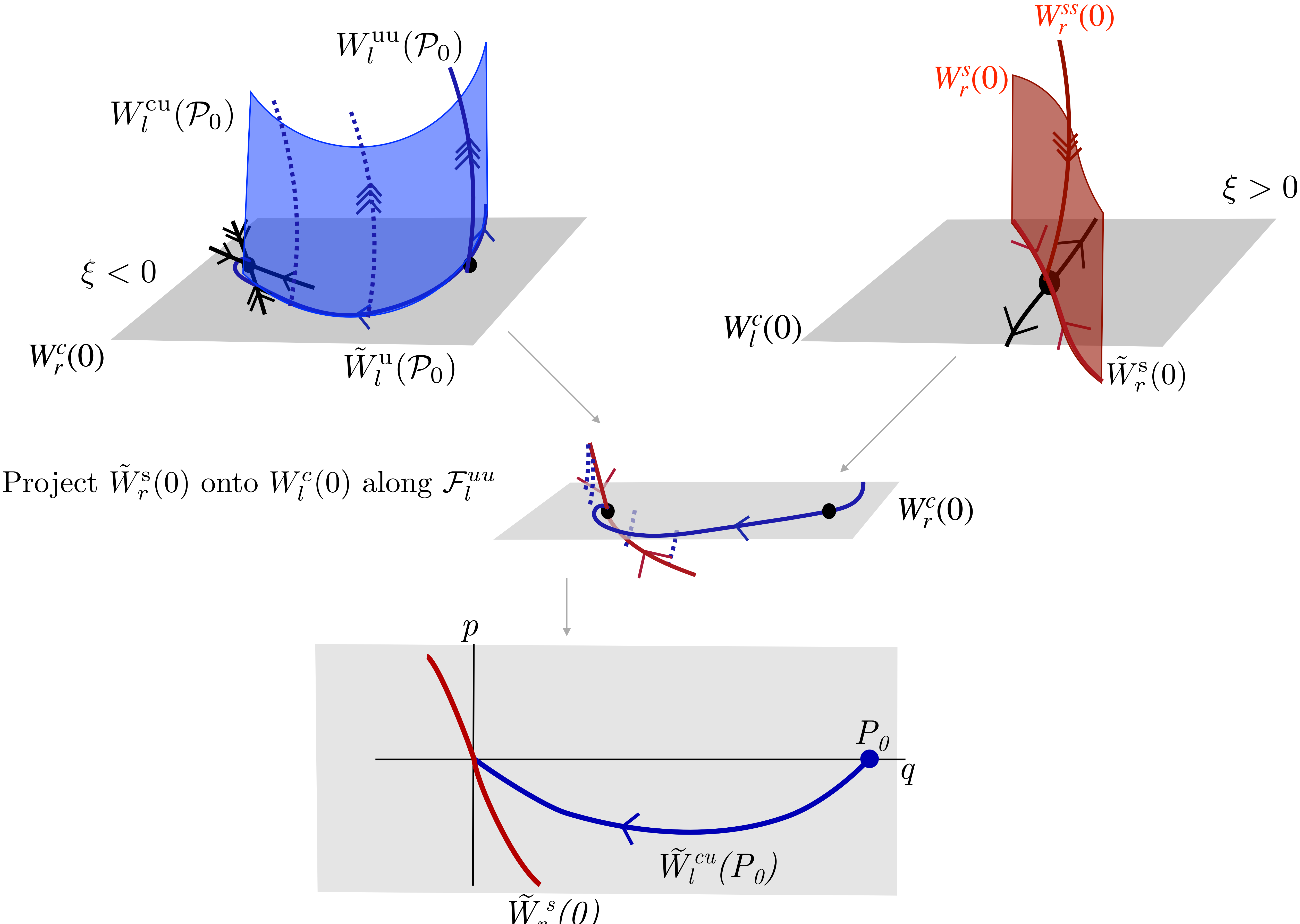
P_0 corresponds to periodic pattern in full dynamics



$$\frac{dq}{d\zeta} = p + \mathcal{O}(\epsilon),$$
$$\frac{dp}{d\zeta} = \frac{1}{4}\left(-q - \tilde{c}p + 3q|q|^2\right) + \tilde{\gamma}(2\mathrm{i}p + \tilde{\gamma}q) + \mathcal{O}(\epsilon)$$

“Project $\tilde{W}_r^s(0)$ onto $W_l^c(0)$ along strong unstable fibers W_l^{uu}”

In more detail:



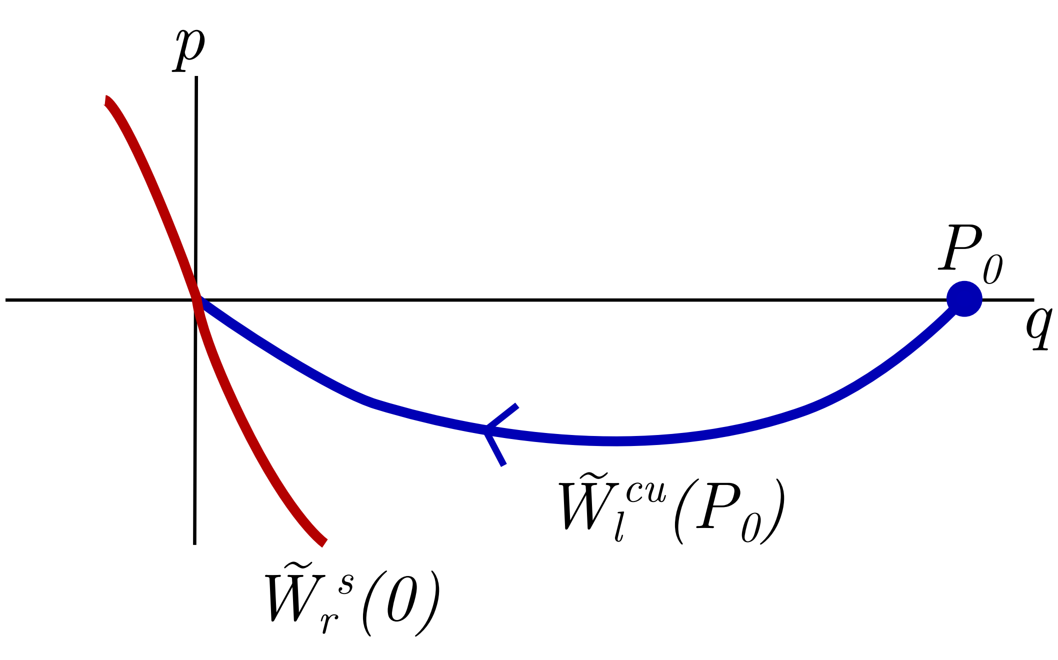
Leading order system: overlay center manifolds

- One finds, to leading order, dynamics governed by the following system:

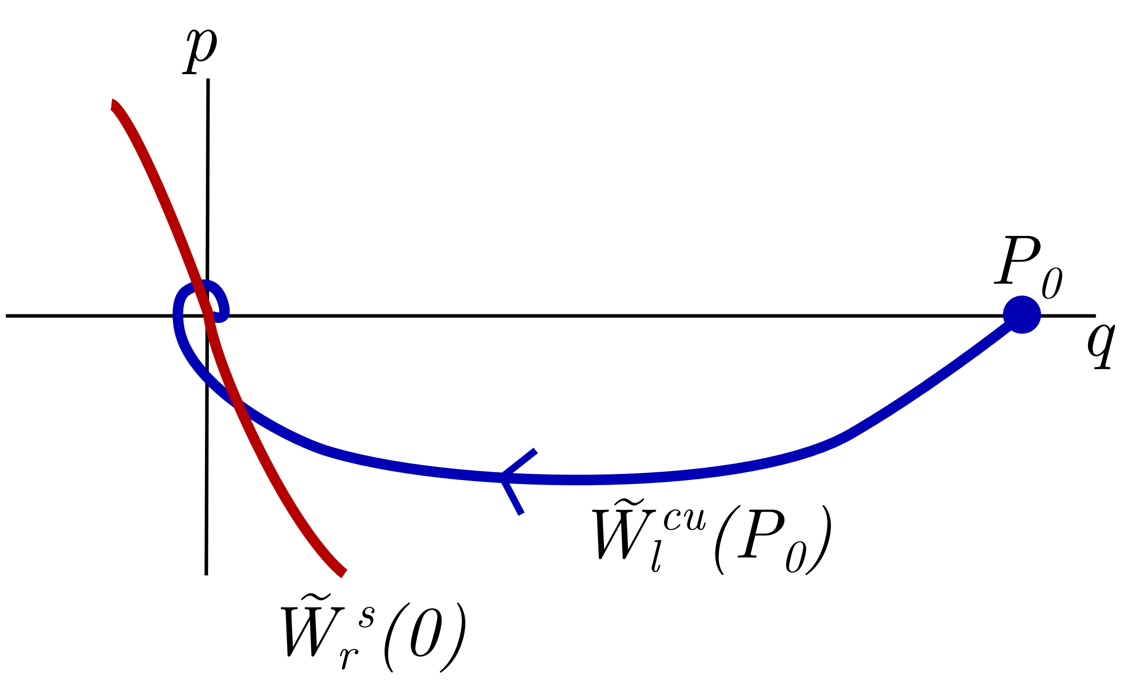
$$\frac{dq}{d\zeta} = p + \mathcal{O}(\epsilon) \quad (p,q) \in \mathbb{C}^2$$
$$\frac{dp}{d\zeta} = \frac{1}{4}\left(-\mathrm{sgn}(-\zeta)q - \tilde{c}p + 3q|q|^2\right) + \tilde{\gamma}(2\mathrm{i}p + \tilde{\gamma}q) + \mathcal{O}(\epsilon),$$

Real subspace $\quad p,q \in \mathbb{R} \qquad \tilde{\gamma} = 0 \qquad \epsilon = 0$

$\tilde{c} > 4$



$\tilde{c} \lesssim 4$



Melnikov integrals and transverse unfolding

- Does the leading order intersection persist for $\tilde{\gamma}, \epsilon \neq 0$?
- Intersection in real subsystem implies 1-D family of intersections, parameterized by rotations, connecting $\tilde{W}^s_r(0)$ and $\mathcal{P} := \{\mathrm{e}^{\mathrm{i}\theta}\mathcal{P}_0 \,:\, (p,q) \mapsto \mathrm{e}^{\mathrm{i}\theta}(p,q)\}$

$$\dim \tilde{W}^s_r(0) \cap \tilde{W}^{cu}_l(\mathcal{P}) = 1 \implies \text{non-transverse intersection}$$

- Look for *Transverse unfolding:* append the equation $\tilde{\gamma}' = 0$, study extended manifolds in $\mathbb{R} \times \mathbb{C}^2$

$$\tilde{W}^s_{r,ext}(I \times 0) = \{(\tilde{\gamma},(p,q)) : \tilde{\gamma} \in I, (q,p) \in \tilde{W}^s_r(0)\}, \quad \tilde{W}^{cu}_{l,ext}(I \times \mathcal{P}) = \{(\tilde{\gamma},(p,q)) : \tilde{\gamma} \in I, (q,p) \in \tilde{W}^{cu}_l(\mathcal{P})\}$$

- Showing non-vanishing of Melnikov integral, (with derivative in $\tilde{\gamma}$), implies transverse intersection of $\tilde{W}^s_{r,ext}(I \times 0) \cap \tilde{W}^{cu}_{l,ext}(I \times \mathcal{P})$
- Can conclude persistence of heteroclinic orbit, for $\epsilon, \tilde{\gamma}$ perturbations

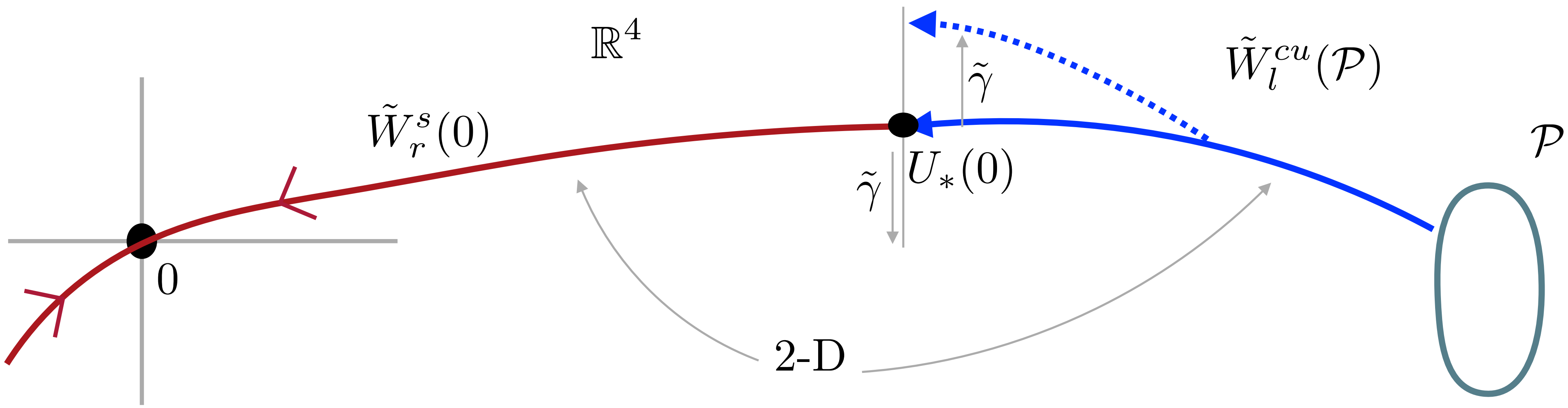
Write as a real system: $q = q_r + \mathrm{i}q_i, p = p_r + \mathrm{i}p_i$

$$\dot{q_r} = p_r,$$
$$\dot{p_r} = -\frac{1}{4}(\mu(\xi)q_r + cp_r - 3q_r(q_r^2 + q_i^2)) - 2\gamma p_i,$$
$$\dot{q_i} = p_i$$
$$\dot{p_i} = -\frac{1}{4}(\mu(\xi)q_i + cp_i - 3q_i(q_r^2 + q_i^2)) - 2\gamma p_r,$$

$\Longrightarrow$ $U_\zeta = F(\xi, U; \tilde{c}, \tilde{\gamma})$ with $U \in \mathbb{R}^4$.

Heteroclinic for $\tilde{\gamma} = 0$

$$U_*(\zeta) = (q_*(\zeta), p_*(\zeta), 0, 0)^T$$



Want to track how invariant manifolds split for $\tilde{\gamma} \neq 0$,

Dimension counting implies intersection is non-generic: 2 + 2 - 1 = 3 $\neq$ 4

$\dim_{\mathbb{R}} \tilde{W}_r^{\mathrm{s}}(0) = 2$ $\quad$ $\dim_{\mathbb{R}} \tilde{W}_l^{\mathrm{cu}}(\mathcal{P}) = 2$ $\quad$ $\dim_{\mathbb{R}} \tilde{W}_l^{\mathrm{cu}}(\mathcal{P}) \cap \tilde{W}_r^{\mathrm{s}}(0) = 1$

$$\dim_{\mathbb{R}} \left[T\tilde{W}_r^s(0) + T\tilde{W}_l^{cu}(0) \right] = 3 \neq 4$$

Not transverse!!!!!

Want to study invariant manifolds near heteroclinic U_*

Consider variations about heteroclinic orbit $\quad U = U_* + V$

$$
\begin{aligned}
V_\zeta &= A(\zeta)V + G(\zeta, V; \tilde{c}, \tilde{\gamma}),\\
A(\zeta) &= D_U F(\zeta, U_*(\zeta); \tilde{c}, 0),\\
G(\zeta, V; \tilde{c}, \tilde{\gamma}) &= F(\zeta, U_*(\zeta) + V; \tilde{c}, \tilde{\gamma}) - F(\zeta, U_*(\zeta); \tilde{c}, 0) - A(\zeta)V.
\end{aligned}
$$

Construct exponential dichotomies for linear system: $V_\zeta = A(\zeta)V$,

$\Phi_{\mathrm{r}}^{\mathrm{s/u}}(\zeta, s)$ for $\zeta, s > 0$,
$\Phi_{\mathrm{l}}^{\mathrm{cu/ss}}(\zeta, s)$ for $\zeta, s < 0$

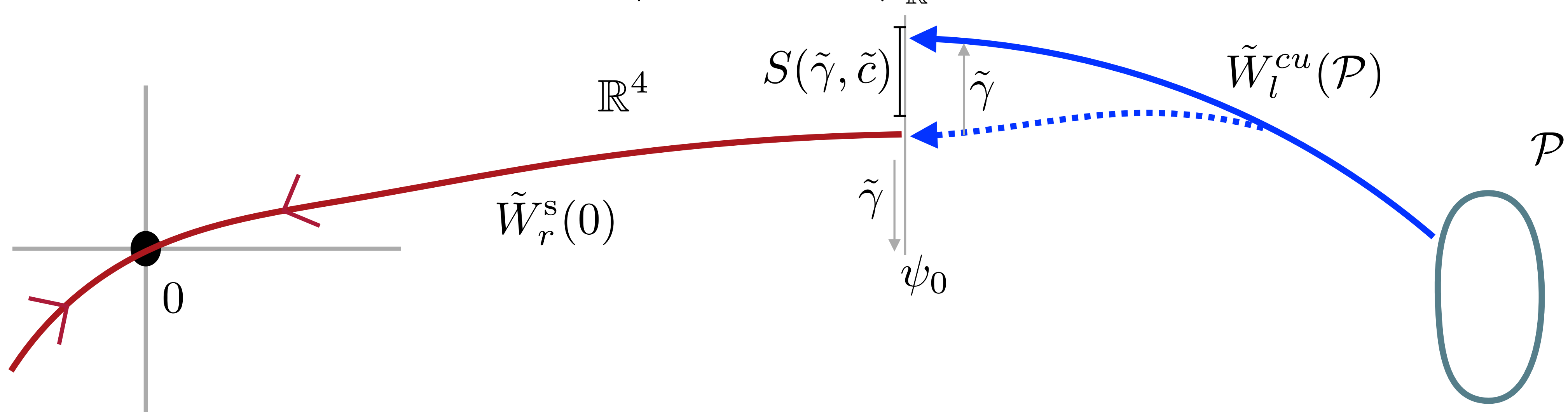
$$
\begin{aligned}
&\tilde{E}_{\mathrm{r}}^{\mathrm{s/u}}(\zeta) := \mathrm{Rg}\Phi_{\mathrm{r}}^{\mathrm{s/u}}(\zeta,\zeta), \quad \tilde{E}_{\mathrm{r}}^{\mathrm{s/u}}(\zeta) = T_{U^*(\zeta)}\tilde{W}_{\mathrm{r}}^{\mathrm{s/u}}(0), \quad \zeta \geq 0,\\
&\tilde{E}_{\mathrm{l}}^{\mathrm{ss/cu}}(\zeta) := \mathrm{Rg}\Phi_{\mathrm{l}}^{\mathrm{ss/cu}}(\zeta,\zeta) \quad \tilde{E}_{\mathrm{r}}^{\mathrm{ss/cu}}(\zeta) = T_{U^*(\zeta)}\tilde{W}_{\mathrm{l}}^{\mathrm{ss/cu}}(\mathcal{P}_0), \quad \zeta \leq 0.
\end{aligned}
$$

Can describe invariant manifolds as graphs over dichotomies, in a neighborhood of $U_*(\zeta)$:

$$
\begin{aligned}
\tilde{W}_{\mathrm{l}}^{\mathrm{cu}}(\mathcal{P}_0) &= \{U^*(0) + v_{\mathrm{l}} + \tilde{h}_{\mathrm{l}}^{\mathrm{cu}}(v_{\mathrm{l}}, \tilde{\gamma}) \,|\, \tilde{h}_{\mathrm{l}}^{\mathrm{cu}} : \tilde{E}_{\mathrm{l}}^{\mathrm{cu}}(0) \times \mathbb{R} \to \tilde{E}_{\mathrm{l}}^{\mathrm{ss}}(0)\},\\
\tilde{W}_{\mathrm{r}}^{\mathrm{s}}(0) &:= \{U^*(0) + v_{\mathrm{r}} + \tilde{h}_{\mathrm{r}}^{\mathrm{s}}(v_{\mathrm{r}}, \tilde{\gamma}) \,|\, \tilde{h}_{\mathrm{r}}^{\mathrm{s}} : \tilde{E}_{\mathrm{r}}^{\mathrm{s}}(0) \times \mathbb{R} \to \tilde{E}_{\mathrm{r}}^{\mathrm{u}}(0)\}.
\end{aligned}
$$

$$
\dim_{\mathbb{R}^4} \tilde{E}_{\mathrm{r}}^{\mathrm{s}}(0) \cap \tilde{E}_{\mathrm{l}}^{\mathrm{cu}}(0) = 1, \implies \left[\tilde{E}_{\mathrm{r}}^{\mathrm{s}}(0) + \tilde{E}_{\mathrm{l}}^{\mathrm{cu}}(0)\right]^{\perp} = \mathrm{span}\{\psi_0\}
$$

Define splitting distance: $\quad S(\tilde{\gamma}, \tilde{c}) = \left\langle \tilde{\psi}_0, \tilde{h}_{\mathrm{l}}^{\mathrm{cu}} - \tilde{h}_{\mathrm{r}}^{\mathrm{s}} \right\rangle_{\mathbb{R}^4}$



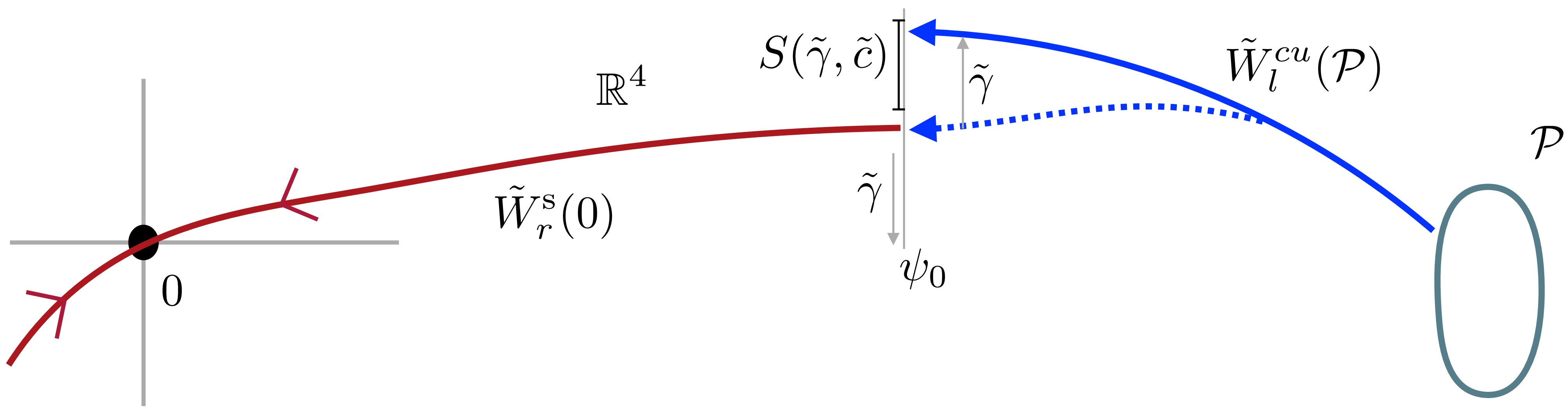
Define splitting distance: $S(\tilde{\gamma}, \tilde{c}) = \left\langle \tilde{\psi}_0, \tilde{h}_{\mathrm{l}}^{\mathrm{cu}} - \tilde{h}_{\mathrm{r}}^{\mathrm{s}} \right\rangle_{\mathbb{R}^4}$

"Melnikov integral"

Can show $\frac{\partial}{\partial\tilde{\gamma}} S(0,0,0) = \left\langle \tilde{\psi}_0, \tilde{h}_{\mathrm{l}}^{\mathrm{cu}}(0;0) - \tilde{h}_{\mathrm{r}}^{\mathrm{s}}(0;0) \right\rangle_{\mathbb{R}^4}$

$$= \left\langle \tilde{\psi}_0, \int_{-\infty}^{0} \Phi_{\mathrm{l}}^{\mathrm{ss}}(0,\zeta)\partial_\gamma G d\zeta - \int_{\infty}^{0} \Phi_{\mathrm{r}}^{\mathrm{u}}(0,\zeta)\partial_\gamma G d\zeta \right\rangle_{\mathbb{R}^4} \neq 0$$

(... using adjoint variational equation: $\psi_\zeta = A(\zeta)^* \psi$)



$\partial_{\tilde{\gamma}} S \neq 0 \implies$ Invariant manifolds split with non-zero "speed" in $\tilde{\gamma}$

Can conclude that invariant manifolds in extended system intersect transversely

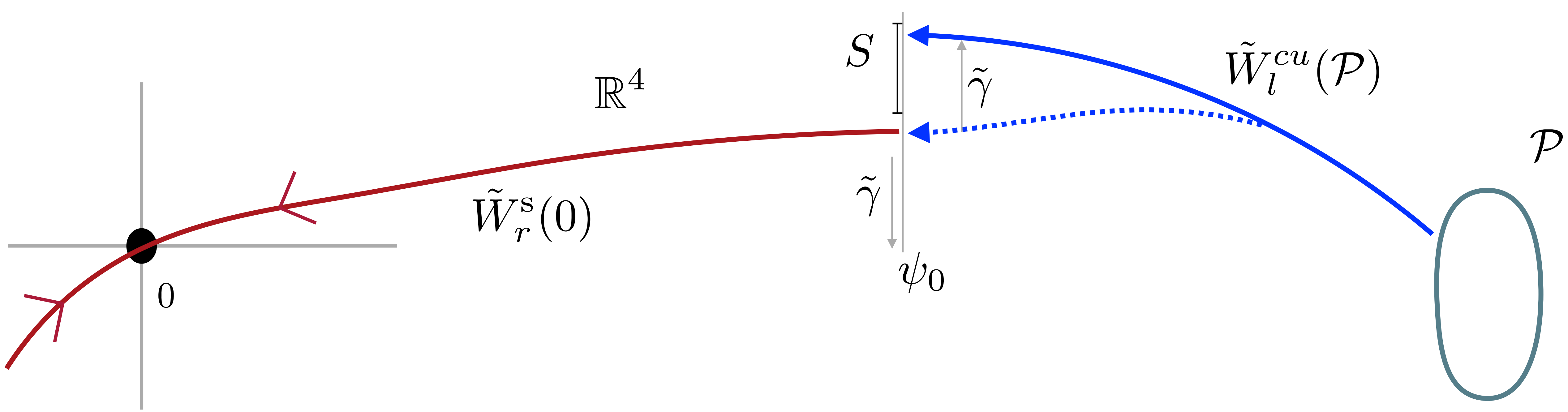
$$\frac{d}{d\zeta} V = A(\zeta) V + G(\zeta, V; \tilde{c}, \tilde{\gamma}) \qquad \tilde{W}^s_{r,ext}(I \times 0) = \{(\tilde{\gamma}, (p,q)) : \tilde{\gamma} \in I, (q,p) \in \tilde{W}^s_r(0)\},$$

$$\frac{d}{d\zeta} \tilde{\gamma} = 0 \qquad \tilde{W}^{cu}_{l,ext}(I \times \mathcal{P}) = \{(\tilde{\gamma}, (p,q)) : \tilde{\gamma} \in I, (q,p) \in \tilde{W}^{cu}_l(\mathcal{P})\}$$

Hence, under $\epsilon, \tilde{c}$-perturbations one can find a $\tilde{\gamma}$ nearby with an intersection

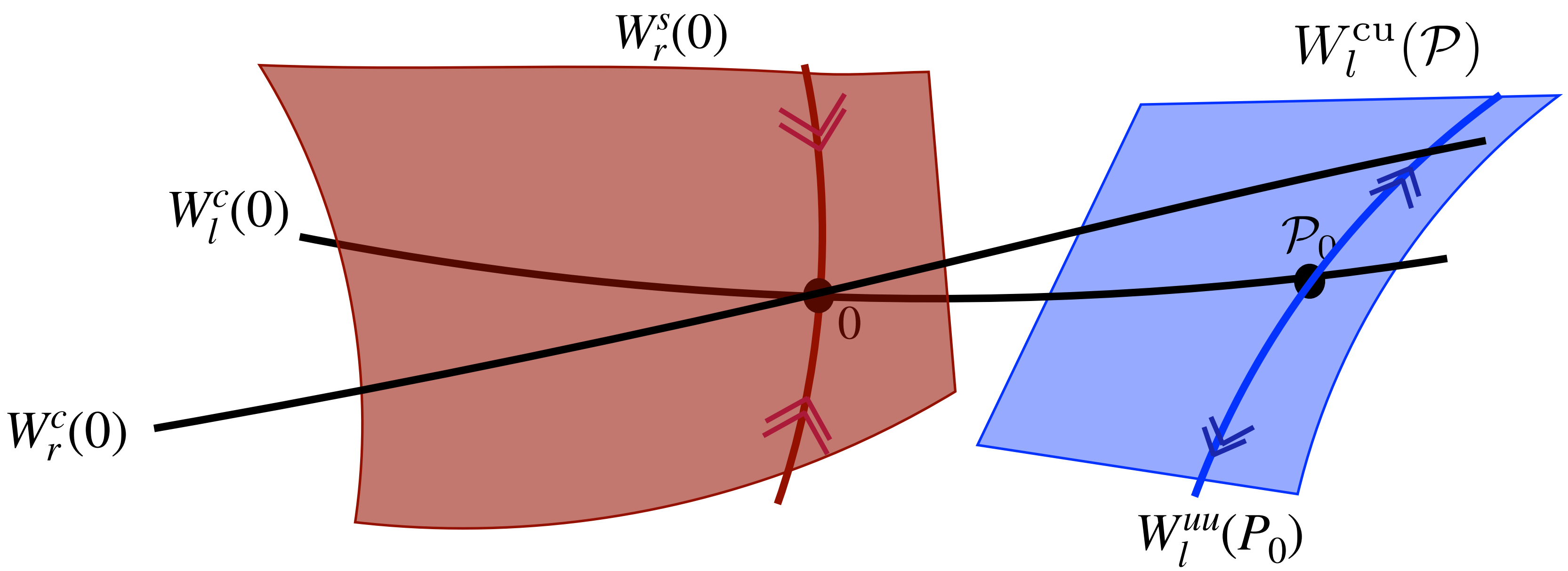
Functional analytic view point:

$\partial_{\tilde{\gamma}} S \neq 0 \implies$ Implicit function thm: Solve $S(\tilde{\gamma}, \tilde{c}) = 0$ for $\tilde{\gamma}$ near $(0, \tilde{c})$

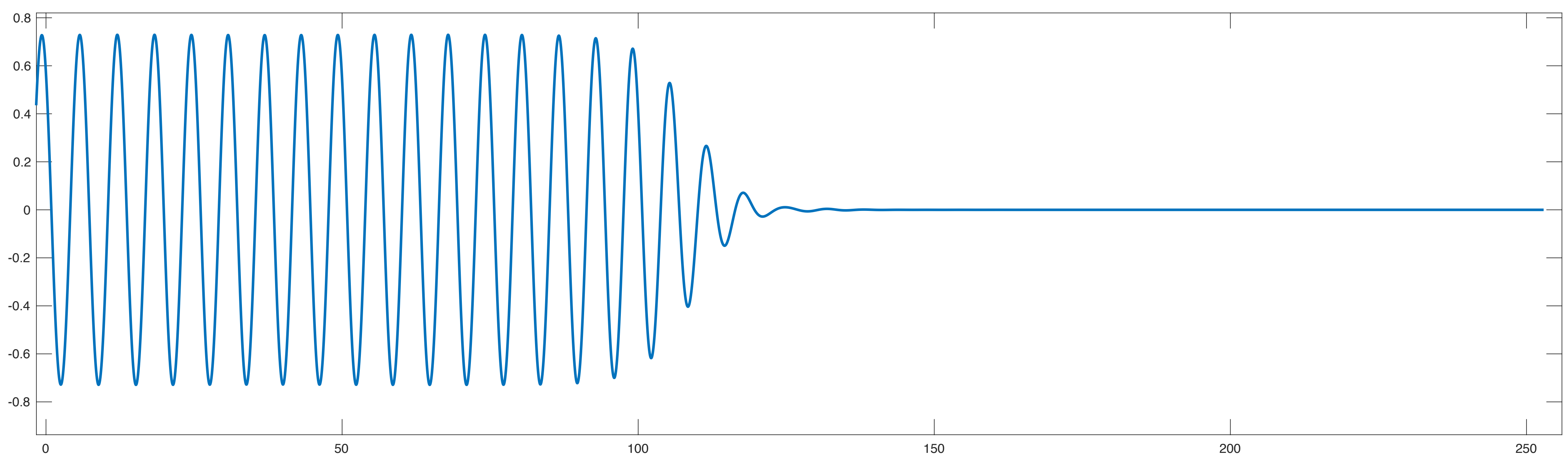


Conclude existence

- Use foliations to lift intersection in full system $W_r^s(0) \cap W_l^{cu}(\mathcal{P}) \neq \varnothing$

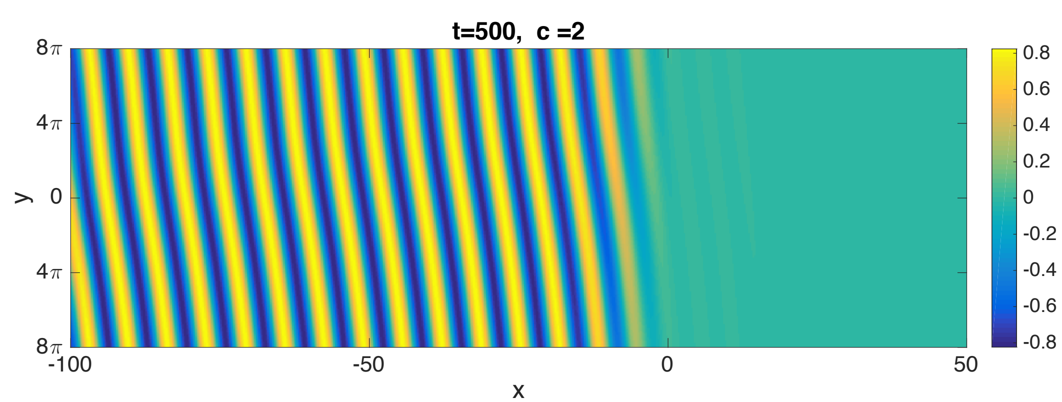


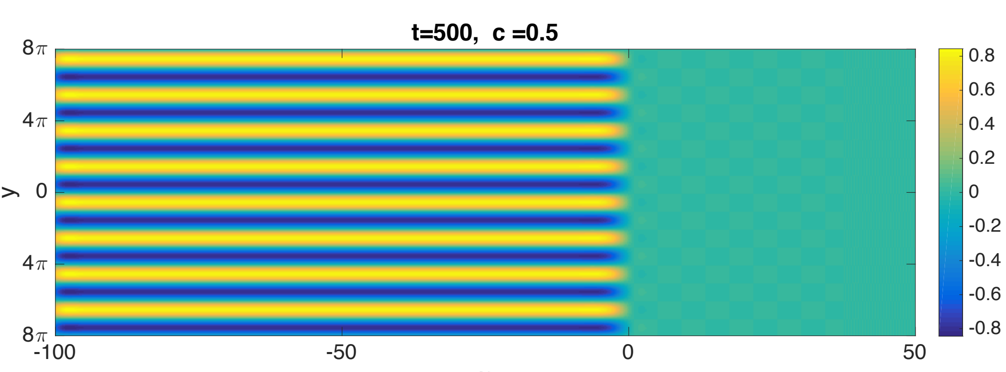
- Existence of pattern forming front to PDE with wavenumber $1 + \epsilon\tilde{\gamma}(\tilde{c})$

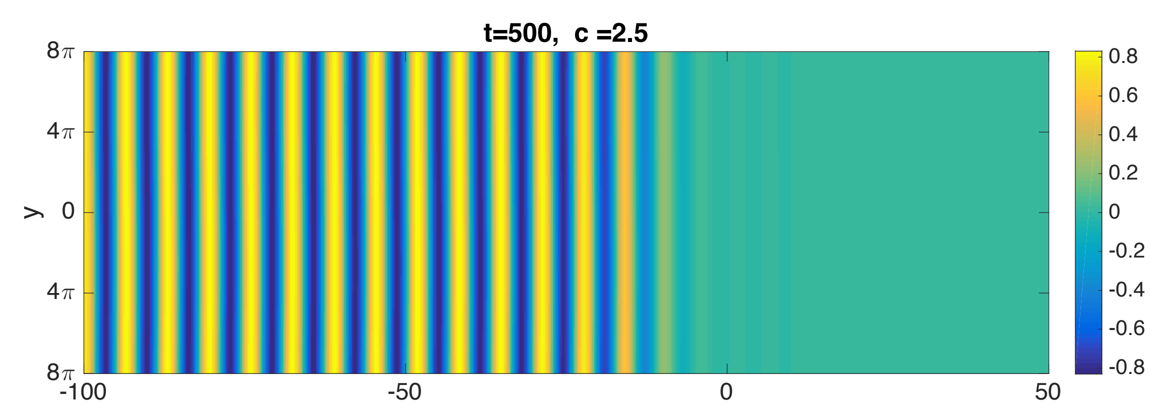


A little bit about 2-D phenomenon

What orientations and wavenumbers of stripes are selected for each quenching speed?







Oblique Stripes

[RG, Scheel; to appear]

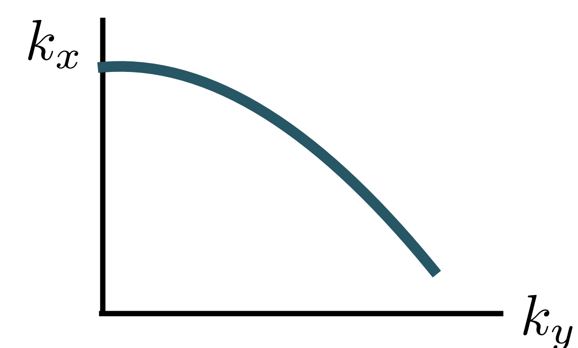
$$u_t = -(1+\Delta)^2 u + \mu(x-ct)u - u^3, \quad (x,y) \in \mathbb{R}^2, t \in \mathbb{R},$$
$$\mu(\xi) = -\mu_0 \operatorname{sgn}(\xi).$$

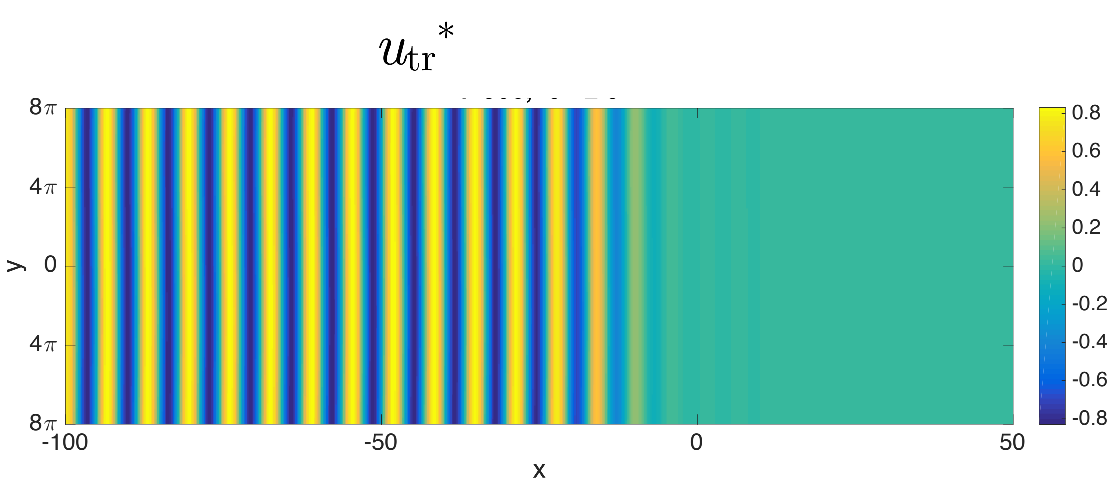
- Homog. State $u = 0$ is stable unstable for $x - ct \gtrless 0$
- $\mu \equiv \mu_0$ system possesses family of roll solutions $u_p(k_x x; k_x)$, $u_p(\theta; k_x) = u_p(\theta + 2\pi; k_x)$

<u>Thm</u>: Assume 1-D pattern exists with c > 0 with wavenumber $k_x{}^*$ and is "generic," then there exists slanted pattern nearby with transverse wavenumber k_y~ 0 with

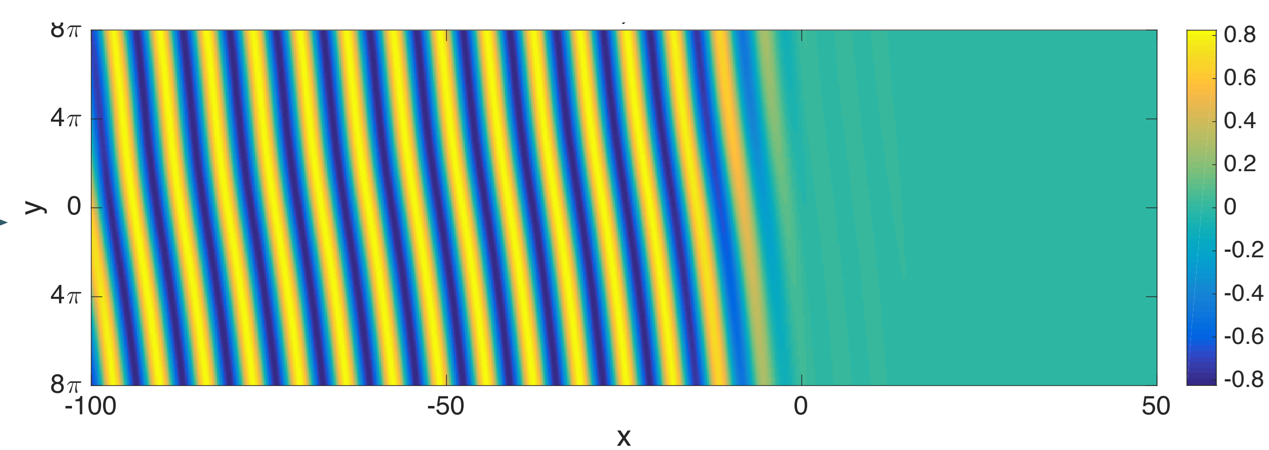
"Angle-selection"

$$k_x(k_y) = k_x^* + d\, k_y^2 + \mathcal{O}(k_y^4)$$





Parallel stripes



Oblique stripes

$$u_p(k_x x + k_y y; |k|), \quad |k|^2 = k_x^2 + k_y^2$$

Note: no onset condition required.

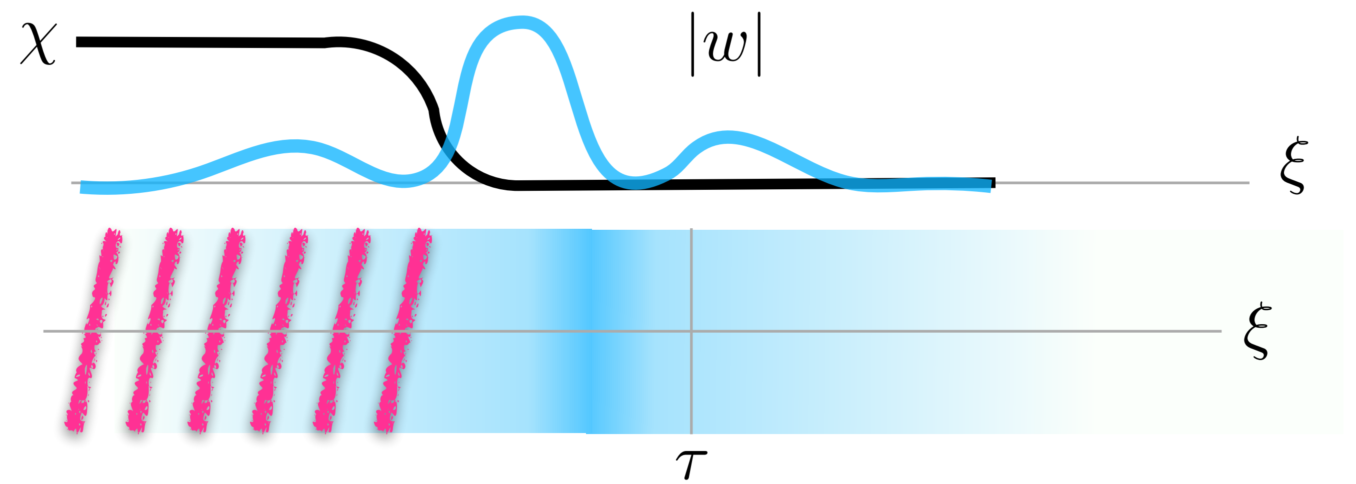
Functional analytic approach

- Look for solutions $u = u(k_x(x-ct), k_y y - \omega t) =: u(\zeta,\tau)$

$$0 = -(1+(k_x\partial_\zeta)^2 + (k_y\partial_\tau)^2)^2 u + \mu(\zeta)u - u^3 + cu_\zeta + \omega u_\tau \quad (\zeta,\tau)\in\mathbb{R}\times\mathbb{T}$$

- Core/Far-field decomposition

$$u = w(\zeta,\tau) + u^*_{\mathrm{tr}}(\zeta,\tau;\mathbf{k}^*) + \chi(\zeta)[u_p(\zeta+\tau;|\mathbf{k}|) - u_p(\zeta+\tau;k_x^*)],$$



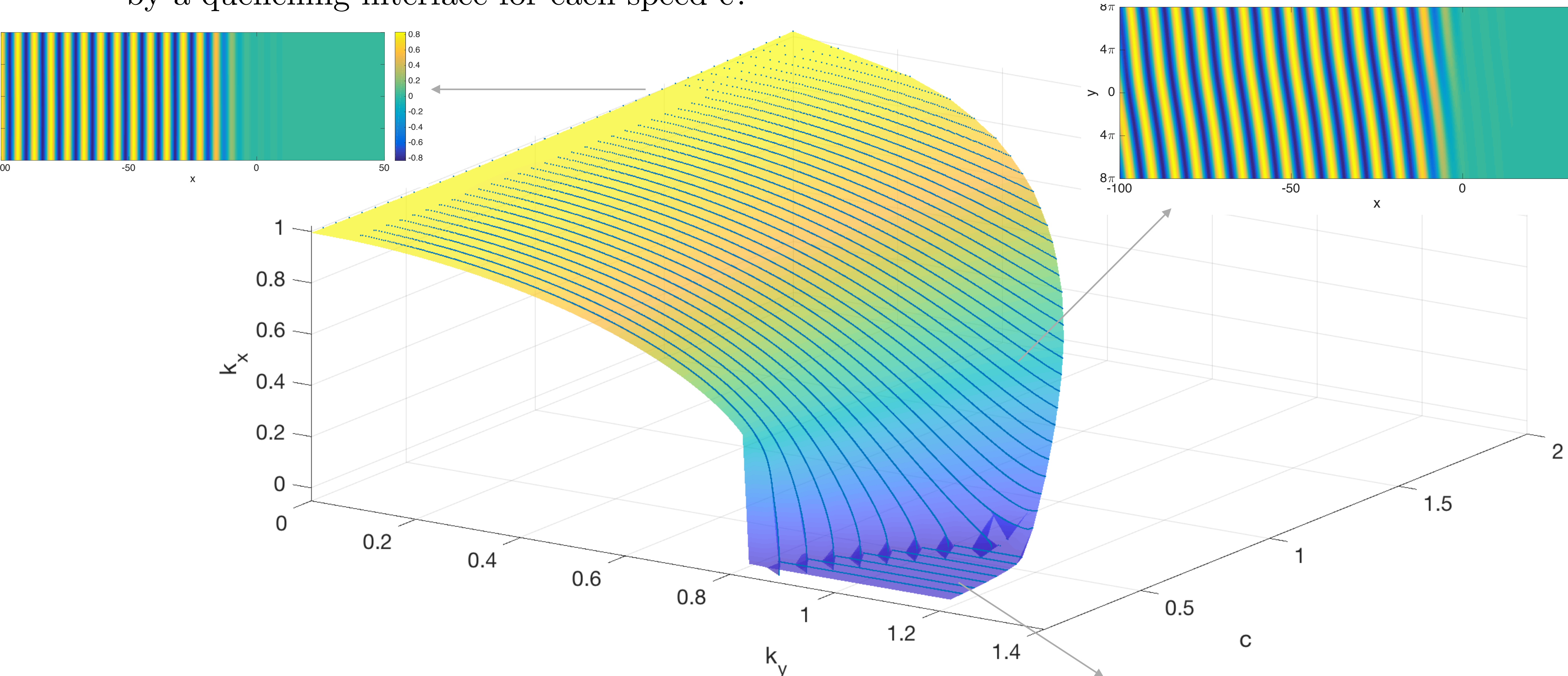
- Insert into equation, obtain nonlinear operator, want to perturb from $(w,k_x,k_y) = (0,k_x^*,0)$

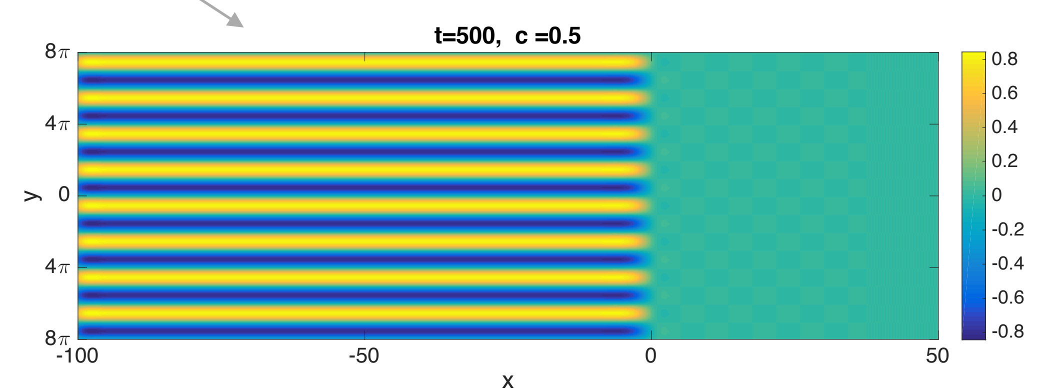
- Use exponential weights to recover Fredholm properties

- Use preconditioning Fourier multiplier to regularized the singular limit k_y ->0.

“Moduli” spaces - numerical continuation

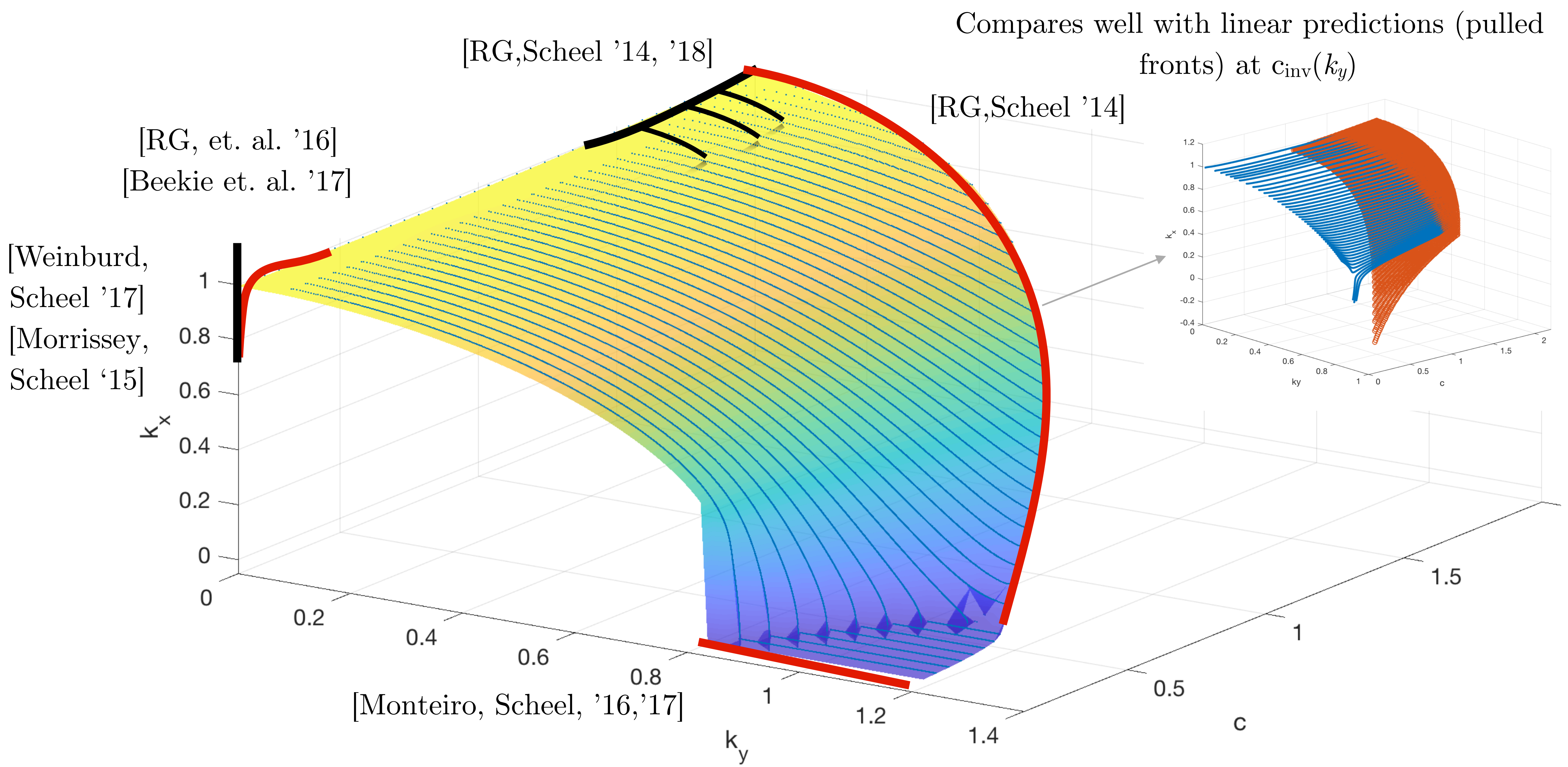
- What orientations and wavenumbers of stripes, parameterized by $k = (k_x, k_y)$ can be selected by a quenching interface for each speed c?





Each point on surfaces represents a striped pattern

“Moduli” spaces - progress



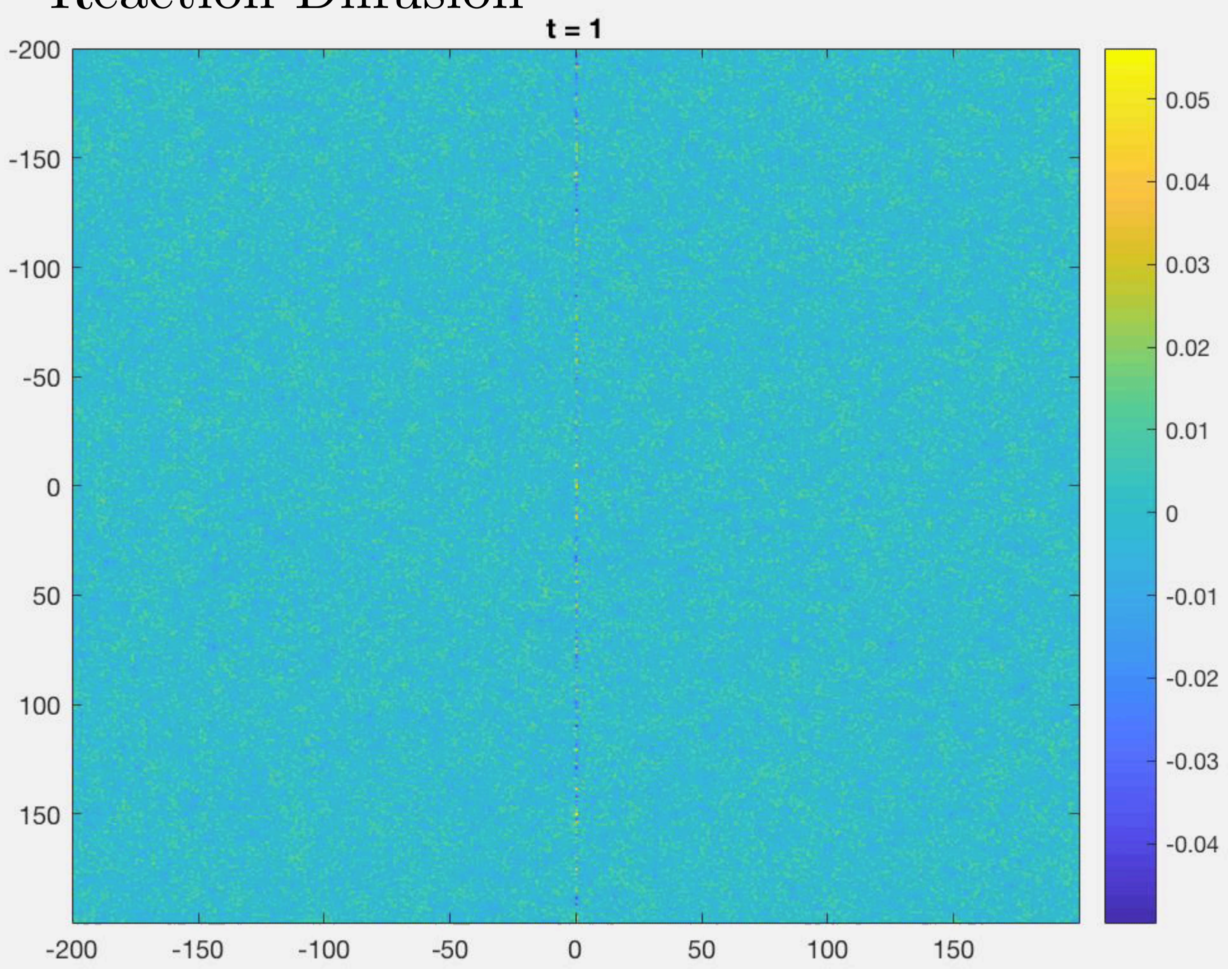
Rigorous results

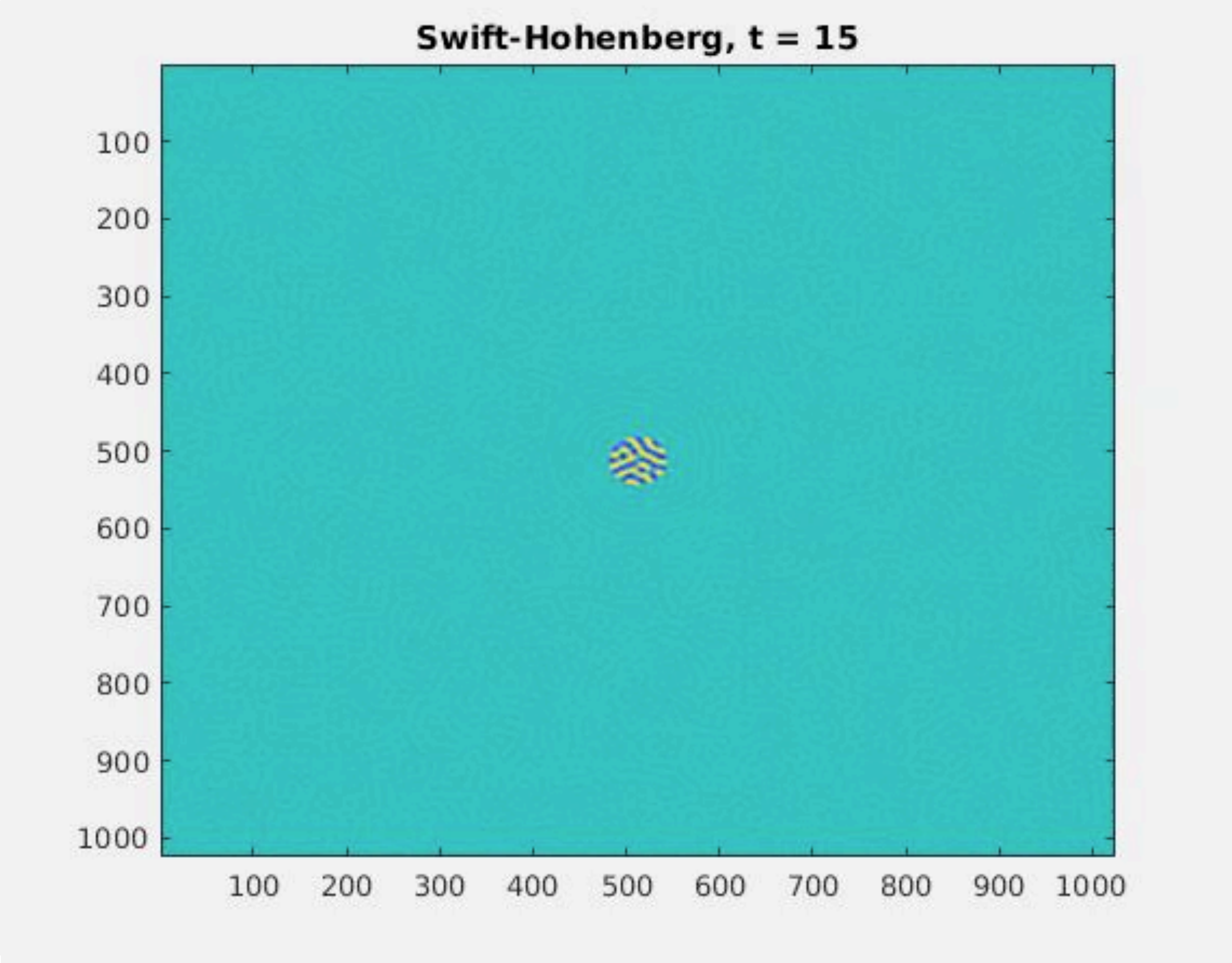
Formal predictions or results in other systems

Other interesting topics

- Other systems: Reaction-Diffusion, Cahn-Hilliard, etc…
- Stability of these patterns
- Modulational equations/dynamics
- Other types of patterns: hexagons, zig-zags, non-planar interfaces

Reaction-Diffusion





Summary

- Growth/quenching mechanisms are an interesting way to mediate patterns in nature
- Mathematics can help:
  - Dynamical systems theory (center-manifold, heteroclinic bifurcation theory, Melnikov integrals) gives powerful tools to illuminate the underlying structure/mechanisms of pattern formation in these models
- There is much more to be done, using a variety of tools and approaches:
  - Rigorous theorems
  - Formal asymptotics
  - Numerical continuation

Thanks!

References:

- Eckmann, J.P., Wayne, C.E., *Propagating fronts and the center manifold theorem, Comm. Math. Phys., 1991*
- RG, Scheel, A. *Pattern-forming fronts in a Swift-Hohenberg equation with directional quenching — parallel and oblique stripes*, J. Lon. Math. Soc., to appear.

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