Superdiffusion of energy in a chain of harmonic oscillators with noise in a magnetic field

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BU/KU-Workshop 2018

June 25, 2018

Overview

In this talk, we consider superdiffusive behavior of energy in 1-dimensional chain of oscillators. This problem is origineted from statistical mechanics.

Today, we have three sections.

- First we introduce a chain of oscillators.
- Second we explain anomalous heat transport and corresponding superdiffusion of energy in a chain of oscillators. We also introduce some conjectures from statistical mechanics and rigorous results about this topic.
- Finally, we introduce our new model, called a chain of harmonic oscillators with noise in a magnetic field, and our main result.

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Introduction

- **First we introduce a chain of oscillators.**

1-dimensional chain $\{p_x(t), q_x(t)\}_x$

Let $\mathbf{S} = \mathbb{Z}, \{1, 2, \cdots, N\}, \mathbb{T}_N$, where $\mathbb{T}_N = \mathbb{Z}/N\mathbb{Z}$.
A chain of oscillators is a Hamiltonian dynamics $\{p_x(t), q_x(t); x \in \mathbf{S}, t \geq 0\}, \{p_x(t), q_x(t)\}_{x \in \mathbf{S}} \in (\mathbb{R} \times \mathbb{R})^{\mathbf{S}}$ defined as

$$\begin{cases} \frac{d}{dt} q_x(t) & = \partial_{p_x} \mathcal{H}(p(t), q(t)) \\ \frac{d}{dt} p_x(t) & = -\partial_{q_x} \mathcal{H}(p(t), q(t)), \end{cases}$$

where

$$\mathcal{H}(p, q) = \sum_{x \in \mathbf{S}} \frac{1}{2} |p_x|^2 + V(q_x - q_{x+1})$$

and $V \in C^{\infty}(\mathbb{R})$ is an interaction-potential.
$p_x(t)$:the momentum of the particle x at time t
$q_x(t)$:the position of the particle x at time t

Introduction

- First we introduce a chain of oscillators.
- **By choosing specific $V(y)$, we can obtain some important chains. Especially, we are interested in**
  $V(y) = \frac{1}{2}y^2$ **: harmonic (linear) chain**

Harmonic chain ($V(y) = \frac{1}{2}y^2$)

$$\{p_x(t), q_x(t); x \in \mathbf{S}, t \geq 0\}, \{p_x(t), q_x(t)\}_{x \in \mathbf{S}} \in (\mathbb{R} \times \mathbb{R})^{\mathbf{S}}$$

$$\begin{cases} \frac{d}{dt} q_x(t) & = p_x(t) \\ \frac{d}{dt} p_x(t) & = q_{x+1}(t) + q_{x-1}(t) - 2q_x(t) \\ & = (\Delta q)_x(t) \end{cases}$$

$$\mathcal{H}(p, q) = \sum_{x \in \mathbf{S}} \frac{1}{2}|p_x|^2 + \frac{1}{2}(q_x - q_{x+1})^2$$

If a chain is a linear system, then we say that the chain is harmonic. Harmonic chain is a special case, and chain models are usually nonlinear systems.

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  $V(y) = \frac{1}{2}y^2 + \alpha y^3 + \beta y^4$ **: Fermi-Pasta-Ulam-Chain (FPU-chain)**
  FPU-chains are typical and important anharmonic(nonlinear) systems.

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  FPU-chains are typical and important anharmonic(nonlinear) systems.
- **Chains of oscillators are microscopic models of heat transport in quasi-1-dimensional objects, such as nanowires , nanotubes. They are used to study thermal conduction in microscopic scale.**

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Anomalous heat transport

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Fourier's law

Fourier's law is the fundamental law of the macroscopic heat transport expressed as

$$j(y,t) = -\kappa \partial_y T(y,t), \quad y \in \mathbb{R}, \ t \geq 0,$$

where $j(y,t)$ is the local energy current, $T(y,t)$ is the local temperature, and $\kappa > 0$ is the thermal conductivity. If $e(y,t)$ is the local energy density and $c > 0$ is the heat capacity, then we have

$$\partial_t e(y,t) + \partial_y j(y,t) = 0,$$
$$e(y,t) = cT(y,t), \quad y \in \mathbb{R}, \ t \geq 0.$$

From the above equations, we see that if Fourier's law is established, then the energy diffuses normally:

$$\partial_t e(y,t) = \frac{\kappa}{c} \Delta e(y,t).$$

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- **Then we explain how the anomalous heat transport in FPU-chains were found.**

Anomalous heat transport in FPU-β-chain

Let us consider FPU-β-chains ($\alpha = 0, \beta > 0$) of N particles in contact with thermal reservoirs operating different temperatures, T_H, T_L. Then the thermal conductivity κ_N is defined as

$$\kappa_N = \frac{N < J >}{T_H - T_L},$$

where $< J >$ is the steady state current per site. As $N \to \infty$, $\kappa_N \sim N^a$, $0 < a < 1$. The divergence of the thermal conductivity means the violation of the Fourier's law and the corresponding superdiffusion of energy.

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- **At this time we state our problem.**

Problem

- Our interest is to derive the hydrodynamic limit for energy of anharmonic chains. (to derive the macroscopic behavior of the energy distribution.) We define $e_x(t)$, the energy of particle x at time t as

$$e_x(t) := \frac{|p_x(t)|^2}{2} + \frac{V(q_{x+1}(t) - q_x(t)) + V(q_x(t) - q_{x-1}(t))}{2}.$$

- We consider the space-time scaling limit for the empirical measure of the energy $\sum_x e_x(t)\delta_x$:

$$\lim_{\epsilon \to 0} \epsilon \sum_{x \in \mathbb{Z}} e_x(\frac{t}{\epsilon^a})\delta_{\epsilon x} = \exists \mathbf{e}(y,t)dy$$

  We want to detect the ratio of the space-time scaling $a > 0$ to get the non-trivial limit, and characterize the macroscopic energy distribution $\{\mathbf{e}(y,t); y \in \mathbb{R}, t \geq 0\}$.

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- Now we review Fourier's law.
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- At this time we state our problem.
- **We also introduce some conjectures about this problem from statistical mechanics.**

Conjectures from statistical mechanics

- There are two universality classes corresponding to the interaction-potential $V(y)$. [Spohn-14]
  V:even $(V(y) = V(-y)) \to a = \frac{3}{2}$, $\partial_t \mathbf{e}(y,t) = -(-\Delta)^{\frac{3}{4}}\mathbf{e}(y,t)$
  V:not even $\to a = \frac{5}{3}$, $\partial_t \mathbf{e}(y,t) = -(-\Delta)^{\frac{5}{6}}\mathbf{e}(y,t)$

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There are some conjectures about the necesarry conditions for anomalous heat transport.

- One of the necessary conditions is that there exists more than two conserved quantities. Hamiltonian dynamics conserves total energy, and chains conserve total momentum because chains are translation-invariant.
- Other necessary condition is that systems are low-dimension, 1-dimension or 2-dimension. Generally, anomalous transport occurs only in low-dimensional nonlinear systems.

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- At this time we state our problem.
- We also introduce some conjectures about this problem from statistical mechanics.
- **Finally, we consider how to solve the problem mathematically and show rigorous results about this topic.**

How to solve the problem

- The mathematical analysis of anomalous transport in anharmonic chains is a very difficult task. In general, we can't prove the ergodicity of the systems and it is not possible to perform explicit computation. There is no rigorous result about this topic. (But there are many numerical results.)

How to solve the problem

- The mathematical analysis of anomalous transport in anharmonic chains is a very difficult task. In general, we can't prove the ergodicity of the systems and it is not possible to perform explicit computation. There is no rigorous result about this topic. (But there are many numerical results.)
- **Recently, stochastic chain model (a harmonic chains perturbed by momentum-conservative stochastic noise) is proposed. This model is an approximation of some anharmonic chain. For this model, we can perform explicit computation and we have some rigorous results about the anomalous heat transport.**

Harmonic chain with conservative noise

$$\{p_x(t), q_x(t); x \in \mathbf{S}, t \geq 0\}, \{p_x(t), q_x(t)\}_{x \in \mathbf{S}} \in (\mathbb{R} \times \mathbb{R})^{\mathbf{S}}$$

$$\begin{cases} dq_x(t) & = p_x(t)dt \\ dp_x(t) & = (\Delta q)_x(t)dt + \frac{\gamma}{2}\Delta(4p_x(t) + p_{x+1}(t) + p_{x-1}(t))dt \\ & \quad + \sqrt{\gamma}\sum_{z;|z-x|\leq 1}(Y_x p_x(t))dw_{x+z}, \end{cases}$$

where $\gamma > 0$ is the strength of the noise and $\{w_x(t); x \in \mathbb{Z}, t \geq 0\}$ are I.I.D. $\mathbb{R}$-valued standard Brownian motions. $\{Y_x; x \in \mathbb{Z}\}$ are first-order differential operators defined as

$$\begin{aligned} Y_x := & (p_x - p_{x+1})\partial_{p_{x-1}} + (p_{x+1} - p_{x-1})\partial_{p_x} \\ & + (p_{x-1} - p_x)\partial_{p_{x+1}}. \end{aligned}$$

We approximate nonlinear effect by stochastic noise. The dynamics conserves the total energy and the total momentum. Since this stochastic noise have some mixing property, we can prove that this stochastic chain is ergodic.

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- Recently, stochastic chain model (a harmonic chains perturbed by momentum-conservative stochastic noise) is proposed. **It was rigorously shown that this stochastic chain has diverging thermal conductivity**:

Theorem (Basile-Bernardin-Olla-09)

$$\kappa_N \sim N^{\frac{1}{2}}, \quad N \to \infty.$$

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- Recently, stochastic chain model (a harmonic chains perturbed by momentum-conservative stochastic noise) is proposed. It was rigorously shown that this stochastic chain has diverging thermal conductivity:

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- **Moreover, the superdiffusion of the energy was rigorously shown**:

Superdiffusion of energy

Theorem (Jara-Komorowski-Olla-15)

Suppose that the initial distribution of the dynamics satisfies $\sup_{0<\epsilon<1} \epsilon^2 E_\epsilon[\int_{\mathbb{T}} dk\, |\widehat{e}(k)|^2] < \infty$, *where* $\mathbb{T}$ *is one-dimensional torus and* $\widehat{e}(k) := \sum_{x\in\mathbb{Z}} e^{-2\pi kx} e_x$, $k \in \mathbb{T}$. *In addition, we assume that there exists some* $W_0 \in \mathbb{L}^1(\mathbb{R})$ *such that for any* $J \in C_0^\infty(\mathbb{R})$, *we have*

$$\lim_{\epsilon\to 0} \epsilon \sum_x J(\epsilon x) E_\epsilon[e_x] = \int_{\mathbb{R}} dy J(y) W_0(y).$$

Then for any test functions $J \in C_0^\infty(\mathbb{R} \times [0,\infty))$, *we have*

$$\lim_{\epsilon\to 0} \epsilon \sum_x \int_0^\infty dt\ J(\epsilon x, t)\mathbb{E}_\epsilon[e_x(\frac{t}{\epsilon^{\frac{3}{2}}})] = \int_{\mathbb{R}} dy \int_0^\infty dt\ J(y,t) W(y,t),$$

where $W(y,t)$ *is the solution of the following fractional diffusion equation:*

$$\begin{cases} \partial_t W(y,t) = -C_\gamma(-\Delta)^{\frac{3}{4}} W(y,t) \\ W(y,0) = W_0(y). \end{cases}$$

Summary of above slides

- Roughly speaking, our purpose is to derive fractional diffusion equation from a chain of oscillators.
- From a chain of harmonic oscillators with conservative noise, we have $\frac{3}{4}$-fractional diffusion equation and the exponent $\frac{3}{4}$ is appeared in the conjecture by H.Spohn. Therefore, this stochastic chain model is considered as good approximation of some anharmonic chain.

Summary of above slides

- Roughly speaking, our purpose is to derive fractional diffusion equation from a chain of oscillators.
- From a chain of harmonic oscillators with conservative noise, we have $\frac{3}{4}$-fractional diffusion equation and the exponent $\frac{3}{4}$ is appeared in the conjecture by H.Spohn. Therefore, this stochastic chain model is considered as good approximation of some anharmonic chain.
- **Until recently, the exponent $\frac{5}{6}$ (also appeared in the conjecture) was not derived rigorously from a chain of oscillators. But in [Saito-Sasada-Suda-18+], we derive $\frac{5}{6}$-fractional diffusion equation from a chain of harmonic oscillators with noise in a magnetic field.**

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- **A chain of harmonic oscillators with noise in a magnetic field is introduced in [Tamaki-Sasada-Saito-17].**

Magnetic model[Tamaki-Sasada-Saito-17]

We define a 1-dimensional chain in 2-dimensional space $\{v_x(t), q_x(t); x \in \mathbf{S}, t \geq 0\}$, $\{v_x(t), q_x(t)\}_{x \in \mathbf{S}} \in (\mathbb{R}^2 \times \mathbb{R}^2)^{\mathbf{S}}$ as

$$\begin{cases} dq_x^i(t) & = v_x^i(t)dt \\ dv_x^i(t) & = (\Delta q^i)_x(t) + \delta_{i,1} B v_x^2(t) - \delta_{i,2} B v_x^1(t) + \gamma \Delta v_x^i(t))dt \\ & \quad + \sqrt{\epsilon\gamma} \sum_{z;|z-x|=1} (Y_{x,z} v_x^i(t)) dw_{x,z}, \end{cases}$$

where $B \in \mathbb{R}$ is the strength of the magnetic field, $\gamma > 0$ is the strength of the stochastic noise, and $\{w_{x,z}(t) = w_{z,x}(t); x \in \mathbf{S}, |z-x| = 1, t \geq 0\}$ are I.I.D. $\mathbb{R}$-valued standard Brownian motions. $\{Y_{x,z}; x, z \in \mathbb{Z}\}$ are first-order differential operators defined as

$$Y_{x,z} = (v_z^2 - v_x^2)(\partial_{v_z^1} - \partial_{v^1 x}) - (v_z^1 - v_x^1)(\partial_{v_z^2} - \partial_{v_x^2}).$$

Magnetic model

- Finally, we introduce our new model, called a chain of harmonic oscillators with noise in a magnetic field, and our main result.
- A chain of harmonic oscillators with noise in a magnetic field is introduced in [Tamaki-Sasada-Saito-17]. **In the same paper, the authors showed that this magnetic model has diverging thermal conductivity**:

Theorem (Tamaki-Sasada-Saito-17)

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- A chain of harmonic oscillators with noise in a magnetic field is introduced in [Tamaki-Sasada-Saito-17]. In the same paper, the authors showed that this magnetic model has diverging thermal conductivity:

Theorem (Tamaki-Sasada-Saito-17)

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The limiting behavior of the thermal conductivity is different from the one of the previous model. Therefore, we expect that the macroscopic behavior of energy is also different. Actually, we have the following result:

Superdiffusion of energy in a magnetic field

Theorem (Saito-Sasada-Suda-18+)

Suppose that $\gamma := \epsilon\gamma_0, \gamma_0 > 0$. *Under some initial conditions, the time evolution of the macroscopic energy distribution* $\{\mathbf{e}(y,t); y \in \mathbb{R}, t \geq 0\}$ *is governed by the following* $\frac{5}{6}$*-fractional diffusion equation:*

$$\partial_t \mathbf{e}(y,t) = -C_{\gamma_0,B}(-\Delta)^{\frac{5}{6}}\mathbf{e}(y,t).$$

We will state the assumption correctly (if we have time). Now we give the rough sketch of the proof. We follow the strategy of [Basile-Olla-Spohn-09] and [Jara-komorowski-Olla-09].

Rough sketch of the proof:Step1

- First we define the local spectral density of energy, (the local energy density on $\mathbb{R} \times \mathbb{T}$), $\{\Omega_i^{\epsilon}(y,k,t); y \in \mathbb{R}, k \in \mathbb{T}, t \geq 0, i = 1,2\}$. In physics, Ω_i^{ϵ} is called the Wigner distribution. We can prove that $\Omega_i^{\epsilon}(y,k,\frac{t}{\epsilon})$ converges to $\Omega_i(y,k,t)$, and $\Omega_i(y,k,t)$ satisfies the following Boltzmann equation:

$$(\partial_t \Omega_i)(y,k,t) + \frac{1}{2\pi}(\partial_k X)(k)(\partial_y \Omega_i)(y,k,t) = (\mathcal{C}\Omega)_i(y,k,t),$$
$$(\mathcal{C}\Omega)_i(y,k,t) := \sum_{j=1,2} \int_{\mathbb{T}} dk' \; R(k,i,k',j)(\Omega_j(y,k',t) - \Omega_i(y,k,t)).$$

where $X(k)$ is the dispersion relation. $\sum_{i=1,2} \int_{\mathbb{T}} dk \Omega_i(y,k,t)$ is the macroscopic energy distribution on $\mathbb{R}$ at time t.

Rough sketch of the proof:Step2

- To derive the fractional diffusion equation, we consider the scaling limit for the solution of the Boltzmann equation. Suppose that $\Omega_i^N(y,k,t)$ is the solution of the scaled Boltzmann equation:

$$(\partial_t \Omega_i^N)(y,k,t) + \frac{1}{2\pi N^{\frac{2}{3}}}(\partial_k X)(k)(\partial_y \Omega_i^N)(y,k,t) = (\mathcal{C}\Omega^N)_i(y,k,t)$$

Then we have

$$\lim_{N\to\infty} |\sum_{i=1,2}\int_{\mathbb{T}} dk \Omega_i^N(y,k,Nt) - \bar{\Omega}(y,t)| = 0,$$

where $\bar{\Omega}(y,t)$ is the solution of the $\frac{5}{6}$-fractional diffusion equation.

Discussion about the main theorem

- By 2-step scaling limit, we have the fractional diffusion equation. The point is that we consider the local spectral density instead of the empirical measure. To derive same equation directly by the scaling limit for the empirical measure is our future work.
- Our proof can be applied to derive the limiting equation from other Hamiltonian systems with some energy-conservative external field.
- As we mentioned, a magnetic model is a first rigorous example of $\frac{5}{6}$-superdiffusion class.