# Dimensions and measures of the limit sets of infinite conformal IFSs related to the generalized complex continued fractions

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# Introduction ①

It is well known that if

- $I$ is a finite index set.
- For all $i \in I$, $\phi_i \colon \mathbb{R}^d \to \mathbb{R}^d$ is contractive and similitude (i.e. $^{\exists} r_i \in (0,1)$ s.t. $^{\forall} x, y \in \mathbb{R}^d$, $|\phi_i(x) - \phi_i(y)| = r_i |x - y|$ ).
- $S := \{\phi_i \colon \mathbb{R}^d \to \mathbb{R}^d | i \in I\}$ satisfies Open Set Condition (i.e. $^{\exists} U \subset \mathbb{R}^d$ : open s.t. $^{\forall} i, j \in I \ (i \neq j)$, $\phi_i(U) \subset U$ and $\phi_i(U) \cap \phi_j(U) = \emptyset$).

then,

- $^{\exists 1} K \subset \mathbb{R}^d$ : non-empty compact s.t. $K = \displaystyle\bigcup_{i \in I} \phi_i(K)$.
- The Hausdorff dimension of K is the unique $t \geq 0$ satisfying $\displaystyle\sum_{i \in I} r_i^t = 1$.
- Both the Hausdorff measure of K and the packing measure of K is positive and finite.

# Introduction ②

- But, recently D. Mauldin and M. Urbanski studied "limit sets of conformal iterated function system (for short, CIFS) with infinitely many mappings".
- And, they introduced an example of CIFS with "strange properties" (the Hausdorff measure of the limit set is zero)

  ( D. Mauldin, M. Urbański (1996) ).

- In this talk, we introduce "a family of CIFSs with infinitely many mappings related to complex continued fractions".
- The limit set of each system in the family also has the strange properties and the Hausdorff dimension of the limit set is real analytic and subharmonic function of the parameter.

# Notation

Let $X$ be a non-empty compact subset of $\mathbb{R}^d$.
Let $I$ be an at most countable index set and we set
$S := \{\phi_i \colon X \to X | i \in I\}$.

- we set $I^* := \bigcup_{n=1}^{\infty} I^n$.
- For each $w = w_1 w_2 w_3 \cdots \in I^{\infty} (= I^{\mathbb{N}})$ and $n \in \mathbb{N}$, we set $w|_n := w_1 \cdots w_n$.
- For each $n \in \mathbb{N}_{\geq 1}$ and $w = w_1 w_2 \cdots w_n \in I^n$, we set $\phi_w := \phi_{w_1} \circ \phi_{w_2} \circ \cdots \circ \phi_{w_n}$.
- If $\phi \colon X \to X$ is differentiable, $|\phi'(x)|$ denotes the norm of the derivative of $\phi$ at $x \in X$ with respect to the Euclidean metric on $\mathbb{R}^d$

# IFS $S$ and limit set $J$

## Definition 1

*Let $X \subset \mathbb{R}^d$ be non-empty and compact.*
*We say that $S := \{\phi_i \colon X \to X \colon \text{injective } | i \in I\}$ is an iterated function system (for short, IFS) if there exists $c \in (0,1)$ such that, for all $i \in I$ and for all $x, y \in X$, $|\phi_i(x) - \phi_i(y)| \le c|x - y|$.*

Using the property of IFS, we can define the limit set of IFS.

## Definition 2

*Let $S$ be a IFS. The limit set of $S$ is defined by*

$$J := \bigcup_{w \in I^\infty} \bigcap_{n \in \mathbb{N}} \phi_{w|_n}(X).$$

- $J = \bigcup_{i \in I} \phi_i(J)$. But, in general, $J$ is not compact.

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# Definition of CIFS

## Definition 3

*Let $X \subset \mathbb{R}^d$ be non-empty, compact and connected. Let $I$ be at most countable ($|I| \geq 2$). An IFS $S$ is called a conformal iterated function system (for short, CIFS) if $S$ satisfies the following conditions.*

1. **Confomality :** *there exist $\epsilon > 0$ and $V \subset \mathbb{R}^d$ open and connected such that $X \subset V$ and, for each $i \in I$, $\phi_i$ extends to $C^{1+\epsilon}$diffeo. on $V$ and is conformal on $V$.*
2. **Open Set Condition (OSC):** *For all $i, j \in I$ $(i \neq j)$, $\phi_i(\mathrm{Int}(X)) \subset \mathrm{Int}(X)$ and $\phi_i(\mathrm{Int}(X)) \cap \phi_j(\mathrm{Int}(X)) = \emptyset$.*
3. **Bounded Distortion Property (BDP):** *there exists $K \geq 1$ such that, for all $x, y \in V$ and for all $w \in I^*$, $|\phi'_w(x)| \leq K \cdot |\phi'_w(y)|$.*
4. **Cone Condition :** *for all $x \in \partial X$, there exists $Con(x, u, \alpha)$: open cone with a vertex $x$, a direction $u$, an altitude$|u|$ and an angle $\alpha$ such that $Con(x, u, \alpha) \subset \mathrm{Int}(X)$.*

# IFS $S$ and limit set $J$

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# Pressure functions

## Definition 4

*Let $S$ be a CIFS. The pressure function of $S$ is defined by*

$$P(t) := \lim_{n\to\infty} \frac{1}{n} \log \sum_{w\in I^n} ||\phi_w'||^t \ \in (-\infty, \infty] \quad (t \geq 0),$$

*where* $||\phi_w'|| := \sup_{x\in X} \sup_{h\in\mathbb{R}^d, |h|=1} |\phi_w'(x)h|$.

- $P(0) = \log |I|$.
- If ${}^{\exists}t_0 \geq 0$ s.t. $P(t_0) < \infty$, then ${}^{\forall}t \geq t_0 \ \ P(t) < \infty$.
- $P$ is strictly decreasing and continuous on $\{t \geq 0 | P(t) < \infty\}$.
- $\lim_{t\to\infty} P(t) = -\infty$.

# Graph of pressure functions

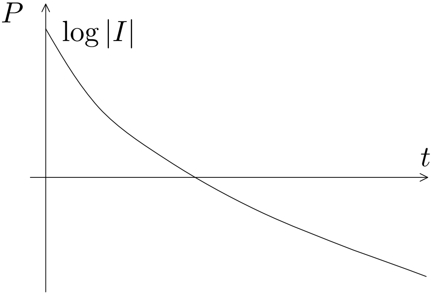
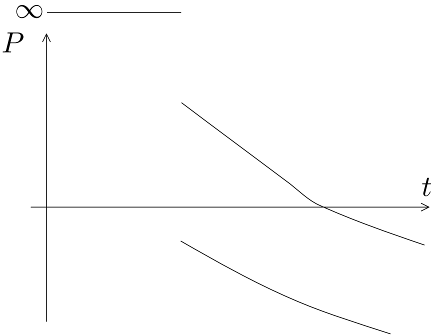



Figure: if $|I|$ is finite.

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In addition,

- If $\phi_i$ is similitude, $P(t) = \log \sum_{i \in I} r_i^t$, ( $r_i$ is similitude ratio of $\phi_i$).

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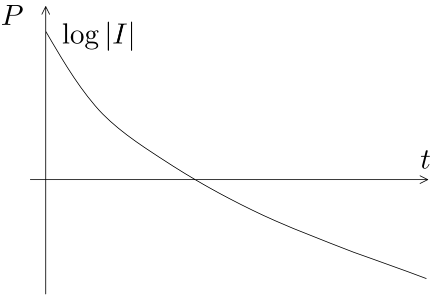


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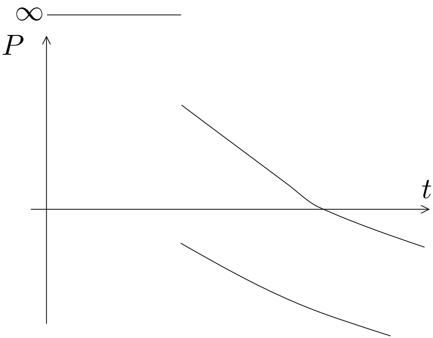


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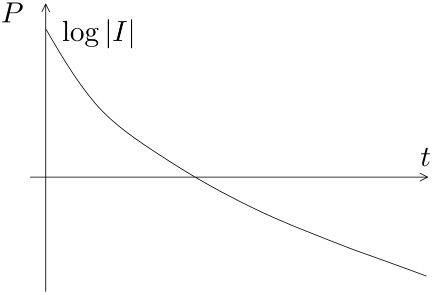


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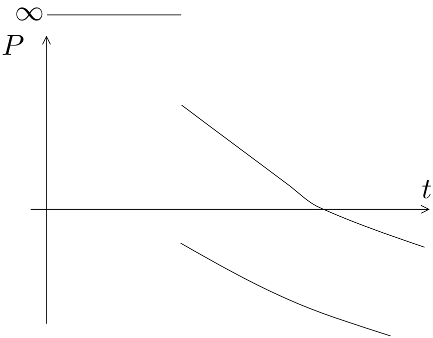


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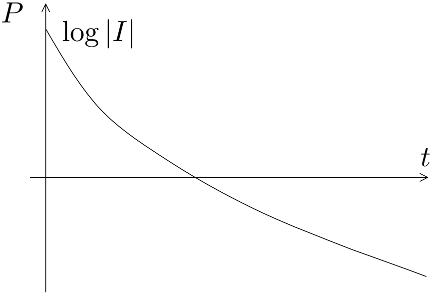
# Known results ①

**Theorem (D. Mauldin, M. Urbański 1996)**

*Let $S$ be a CIFS and let $J$ be the limit set of $S$. Then,*

- *if there exists $t \geq 0$ s.t. $P(t) = 0$, then $\dim_{\mathcal{H}}(J) = t$.*
- *In general, $\dim_{\mathcal{H}}(J) = \inf\{t \geq 0 | \ P(t) < 0\}$.*

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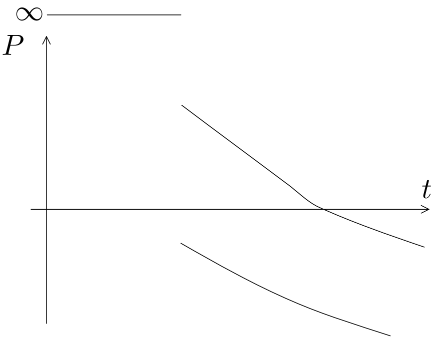


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- We set $X := \{z \in \mathbb{C} || z - 1/2| \leq 1/2\}$.
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# Family of IFSs of generalized complex continued fractions

- We set $A_1 := \{\tau = u + iv \in \mathbb{C} \mid u \geq 0, v \geq 1\}$ (parameter space).
- And, we set $X := \{z \in \mathbb{C} || z - 1/2| \leq 1/2\}$ and $I := \mathbb{N}^2$.

## Definition 5

- *For all $\tau \in A_1$,*
  *we set $S_\tau := \{\phi_{(a,b)}(z) := 1/(z + a + b\tau) | (a,b) \in I\}$ on $X$.*
- *$\{S_\tau\}_{\tau \in A_1}$ is called the family of IFSs of generalized complex continued fractions.*

## Proposition 6

*We have that $\{S_\tau\}_{\tau \in A_1}$ is an "analytic" family of CIFSs with "good properties".*

- We denote by $J_\tau$ the limit set of $S_\tau$ $\quad (\tau \in A_1)$.
- We denote by $h_\tau$ the Hausdorff dimension of $J_\tau$ $\quad (\tau \in A_1)$.

# Main results ①

- We denote by $J_\tau$ the limit set of $S_\tau$ $\quad (\tau \in A_1)$.
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## Theorem A

*Let $\{S_\tau\}_{\tau \in A_1}$ be the family of IFSs of generalized complex continued fractions. Then, we have the followings.*

- *For all $\tau \in A_1$, $1 < h_\tau < 2$.*
- $\displaystyle\lim_{\tau \to \infty, \tau \in A_1} h_\tau = 1$
  *(i.e. for all $\epsilon > 0$, there exists $M > 0$ such that, for all $\tau \in A_1$ with $|\tau| \geq M$, we have $|h_\tau - 1| < \epsilon$).*
- *$\tau \mapsto h_\tau$ is continuous on $A_1$*
  *and $\tau \mapsto h_\tau$ is real-analytic and subharmonic on $\mathrm{Int}(A_1)$.*
  $\rightsquigarrow$*we have* $\max\{h_\tau | \tau \in A_1\} = \max\{h_\tau | \tau \in \partial A_1\}$.

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- We denote by $J_\tau$ the limit set of $S_\tau \quad (\tau \in A_1)$.
- We denote by $h_\tau$ the Hausdorff dimension of $J_\tau \quad (\tau \in A_1)$.
- We denote by $\mathcal{H}^s$ the s-dimensional Hausdorff measure.
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**Theorem B**

*Let $\{S_\tau\}_{\tau \in A_1}$ be the family of IFSs of generalized complex continued fractions. Then, we have the following.*

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# Future works and reference

**Future work**

- Generalization of these CIFSs ( or families of CIFSs ).
  $\rightsquigarrow$ Conformal graph directed markov system.
- Are there any other example with some strange properties?

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