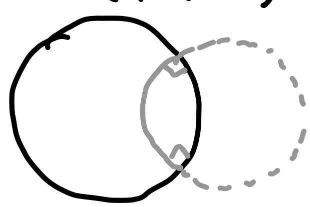
Boston-Keio Workshop 2018, 06/25–06/29

Equivariant harmonic maps and rigidity of discrete groups.   2018/06/27

§1 Nonpositively curved space

$\mathbb{R}^2$

$\mathbb{H}^2 = \left(B^2, \dfrac{4}{(1-|x|^2)^2}\sum_{i=1}^{2} dx_i^2\right)$ $K \equiv -1$ ← sectional curv



(Y, d) a complete metric sp

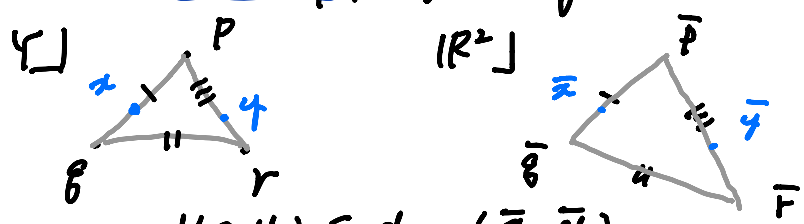
· $I \subset \mathbb{R}$ closed interval

  $c : I \to Y$ is a geodesic $\Longleftrightarrow$ $\forall a, b \in I$ $d(c(a), c(b)) = |a-b|$

· (Y, d) is a geodesic sp

  $\Longleftrightarrow$ $\forall p, q \in Y$ $\exists c : [0, d] \to Y$ s.t. $c(0) = p$, $c(d) = q$
  $d = d(p, q)$

· (Y, d) is a CAT(0) sp (CAT(−1) sp), if its geodesic and



$d(x, y) \leq d_{\mathbb{R}^2}(\bar{x}, \bar{y})$

$\mathbb{H}^2$]

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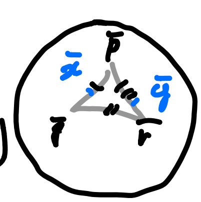
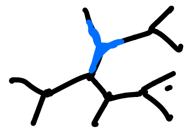


Ex

$K \le -1$ $\quad CAT(-1)$

- a simply connected Riem mfd with $K \le 0$ is $CAT(0)$
- a tree is $CAT(-1)$



- a Euclidean building is $CAT(0)$
- hyperbolic $\qquad CAT(-1)$

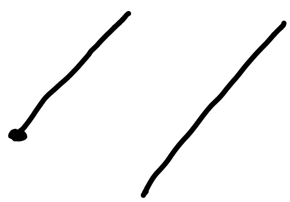
$c, c' : [0, \infty) \to Y$ geodesic rays

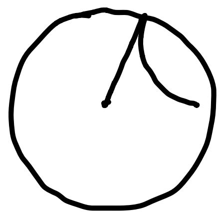
$c \sim c' \iff \exists M \ge 0$ s.t. $\forall t \in [0, \infty)$ $d(c(t), c'(t)) \le M$

asymptotic

$\partial Y = \{ c : [0, \infty) \to Y \text{ geodesics} \} / \sim \quad \ni [c] \leftrightarrow c(\infty)$

$\partial \mathbb{R}^2 = S^1 \qquad \partial \mathbb{H}^2 = S^1$



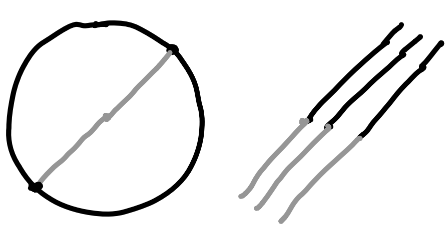


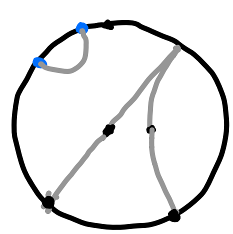
Remark

- Y loc. cpt $\Rightarrow Y \cup \partial Y$ cpt
- The action of $Isom(Y)$ extends to ∂Y

Observation

$\partial \mathbb{R}^2$ and $\partial \mathbb{H}^2$ are very different.





$\forall \xi, \xi' \in \partial \mathbb{H}^2$ can be "joined" by a geodesic in $\mathbb{H}^2$

§2 Energy of equivariant map

Γ a finitely generated group

μ a probability measure on $\Gamma \times \Gamma$ s.t.

- $\mu(\gamma, \gamma') = \mu(\gamma', \gamma)$
- $\mu(\gamma\gamma', \gamma\gamma'') = \mu(\gamma', \gamma'')$
- $\{\gamma \in \Gamma \mid \mu(e, \gamma) \neq 0\}$ generates Γ
- $\# \underline{\qquad\qquad\qquad} < \infty$

) transition probability of "random walk" on Γ

$$\mu^n(\gamma, \gamma') = \sum_{\gamma_1, \cdots, \gamma_{n-1}} \mu(\gamma, \gamma_1)\mu(\gamma_1, \gamma_2) \cdots \mu(\gamma_{n-1}, \gamma')$$

$\rho : \Gamma \to \mathrm{Isom}(Y, d)$ homomorphism

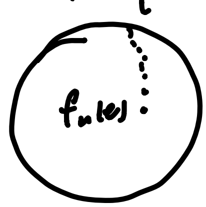
$f : \Gamma \to Y$ ρ-equivariant i.e., $f(\gamma\gamma') = \rho(\gamma) f(\gamma')$

$\rightsquigarrow f(\Gamma) = \rho(\Gamma) f(e)$

- $E_{\mu^n}(f) = \frac{1}{2} \sum_{\gamma \in \Gamma} \mu^n(e, \gamma)\, d(f(e), f(\gamma))^2$ <u>μ^n-energy</u> of f
- f is <u>μ-harmonic</u> $\iff$ f minimizes E_μ

<u>Fact</u> Y loc cpt

$\rho(\Gamma)$ fixes no pt in $\partial Y \Rightarrow \exists f : \Gamma \to Y$ μ-harmonic



One of the origin of this kind of approach is:

Geometric Superrigidity (Mok-Siu-Yeung, Jost-Yau)

Γ as in Cor

Y : CAT(0) manifold. $\rho : \Gamma \to \mathrm{Isom}(Y)$

$\Rightarrow$ (i) $\rho(\Gamma)$ fixes a pt in $Y \cup \partial Y$

(ii) $\exists f : SL(n,\mathbb{R})/SO(n) \longrightarrow Y$ ρ-equiv. isometric embedding

Thm (I)

Y loc cpt CAT(0) sp

$\rho : \Gamma \to \mathrm{Isom}(Y)$, $\rho(\Gamma)$ fixes no pt in $\partial Y \cup Y$

$\forall C \subset \partial Y$ $\rho(\Gamma)$-inv set, $\exists \xi, \xi' \in C$ joined by geod in Y

$\Rightarrow$ (i) $\exists F \subset Y$ $\rho(\Gamma)$-inv flat subsp ($F \cong \mathbb{R}^n$)

(ii) $\liminf\limits_{n\to\infty} \dfrac{E_{\mu^n}(f)}{n^2} > 0$

$\wedge$
∞

Remark

In case (i), μ-harmonic map f satisfies $E_{\mu^n}(f) = n E_\mu(f)$

(This can be a subject of another interest.)

Remark

(ii) $\Rightarrow$ f extends to $\tilde{f} : \partial_\mu \Gamma \to \partial Y$: ρ-equiv

Karlsson-Margulis

Poisson boundary

$\partial_\mu \Gamma = (\partial_\mu \Gamma, \nu) =$ "$\lim\limits_{n\to\infty} (\Gamma, \mu^n(e,\cdot))$"

Cor (Haettel)

$\Gamma < SL(n,\mathbb{R})$ discrete $\Gamma \backslash SL(n,\mathbb{R})$ cpt $n \geq 3$

Y : CAT(-1), $\rho : \Gamma \to \mathrm{Isom}(Y)$

$\Rightarrow \rho(\Gamma)$ fixes a pt in $Y \cup \partial Y$