Variational proof of the existence of brake orbits in the planar 2-center problem

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n-center problem & brake orbits

- n-center problem :

$$\ddot{\boldsymbol{q}} = -\sum_{k=1}^{n} \frac{m_k}{|\boldsymbol{q} - \boldsymbol{a}_k|^3}(\boldsymbol{q} - \boldsymbol{a}_k) \qquad (\boldsymbol{q} \in \mathbb{R}^d)$$

  where $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n \in \mathbb{R}^d$ are constant vectors.

- $\boldsymbol{q}(t)$ is a brake orbit.

  $:\Leftrightarrow {}^{\exists}T_2 > {}^{\exists}T_1 > 0$ such that $\dot{\boldsymbol{q}}(T_1) = \dot{\boldsymbol{q}}(T_2) = 0.$

  (Excluding equilibrium points)

- In the potential systems, brake orbits are $2(T_2 - T_1)$-periodic solutions.

Q. Do brake orbits exist in the planar 2-center problem?

Setting : $m_1 = 1$, $m_2 = m$, $\boldsymbol{a}_1 = \boldsymbol{a} = (1, 0)$, $\boldsymbol{a}_2 = -\boldsymbol{a} = (-1, 0)$.

$\rightarrow$ Variational methods

$$\text{Lagrangean: } L(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \frac{1}{2}|\dot{\boldsymbol{q}}|^2 + \frac{1}{|\boldsymbol{q} - \boldsymbol{a}|} + \frac{m}{|\boldsymbol{q} + \boldsymbol{a}|}$$

$$\text{Action functional: } \mathcal{A}(\boldsymbol{q}) = \int_0^T L(\boldsymbol{q}, \dot{\boldsymbol{q}})dt$$

Boundary conditions : $\boldsymbol{q}(0) \in A := \{(x, 0) \mid -1 \leq x \leq 1\}$, $\boldsymbol{q}(T) \in \mathbb{R}^2$.

$\mathcal{A}'(\boldsymbol{q}) = 0 \Longleftrightarrow \boldsymbol{q}$ is a solution of the planar 2-center problem.

Existence of minimizer

Facts :

- With standard arguments of the existence of a minimizer, there is a minimizer $\boldsymbol{q}^*$ of $\mathcal{A}(\boldsymbol{q})$ satisfying the boundary condition.
- If $\boldsymbol{q}^*$ has no collision, $\dot{\boldsymbol{q}}^*(0) \perp A, \dot{\boldsymbol{q}}^*(T) = 0$ and $\mathcal{A}'(\boldsymbol{q}^*) = 0$ hold.
- If :

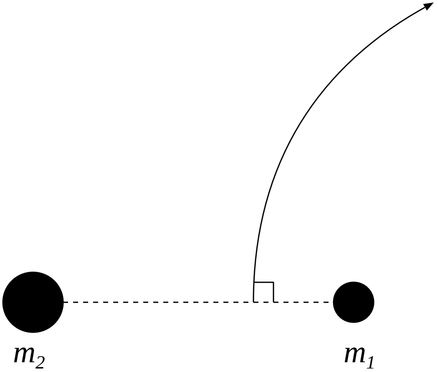
  (Col) $\boldsymbol{q}^*$ is not a collision solution.

  (Eq) $\boldsymbol{q}^*$ is not an equilibrium point.

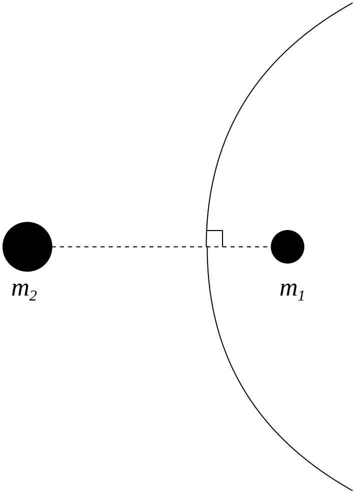
  Then we obtain a $4T$-periodic brake orbit.

Shape of $\boldsymbol{q}^*$

If $\boldsymbol{q}^*$ satisfies (Col) and (Eq), $\boldsymbol{q}^*$ is a part of a brake orbit, i.e. from $t = 0$ to $t = T$ of $4T$-periodic one.



Shape of the whole brake orbit



Under what condition $\boldsymbol{q}^*$ is not an equilibrium point

An equilibrium point : $\boldsymbol{q}_{\mathrm{eq}} = (b, 0)\ \left(b = \dfrac{\sqrt{m}-1}{\sqrt{m}+1}\right)$.

(The second variation)

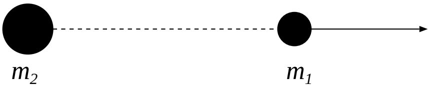
$$\mathcal{A}''(\boldsymbol{q})(\boldsymbol{\delta}) = \lim_{h\to 0}\int (\boldsymbol{\delta}(t), \dot{\boldsymbol{\delta}}(t))\ \nabla^2 L\big|_{(\boldsymbol{q},\dot{\boldsymbol{q}})=(\boldsymbol{q}+h\boldsymbol{\delta},\dot{\boldsymbol{q}}+h\dot{\boldsymbol{\delta}})}\ (\boldsymbol{\delta}(t), \dot{\boldsymbol{\delta}}(t))^{\mathrm{T}} dt$$

$\rightarrow$ If $T > \gamma = \dfrac{\sqrt{2}\pi m^{1/4}}{(1+\sqrt{m})^2}$, $^{\exists}\boldsymbol{\delta}$ such that $\mathcal{A}''(\boldsymbol{q}_{\mathrm{eq}})(\boldsymbol{\delta}) < 0$.

$\rightarrow T > \gamma \Rightarrow \boldsymbol{q}^*$ is not an equilibrium point.

A minimizer in collision solutions

Let $\boldsymbol{q}_{\mathrm{col}}$ be a minimizer in collision solutions.

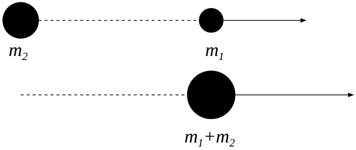


We estimate $\mathcal{A}(\boldsymbol{q}_{\mathrm{col}})$:

$$\begin{aligned}\mathcal{A}(\boldsymbol{q}_{\mathrm{col}}) &= \int_0^T \frac{1}{2}|\dot{\boldsymbol{q}}_{\mathrm{col}}|^2 + \frac{1}{|\boldsymbol{q}_{\mathrm{col}} - \boldsymbol{a}|} dt + \int_0^T \frac{m}{|\boldsymbol{q}_{\mathrm{col}} + \boldsymbol{a}|} dt \\ &\geq \frac{3}{2}\pi^{2/3}T^{1/3} + \int_0^T \frac{m}{|\boldsymbol{q}_{\mathrm{col}} + \boldsymbol{a}|} dt \qquad (1)\end{aligned}$$

Estimate the action of collisions

We estimate the second term in (1) :



$$\int_0^T \frac{m}{|\boldsymbol{q}_{\mathrm{col}} + \boldsymbol{a}|} dt = \int_0^T \frac{m}{q_1(t)+1} dt > \frac{m}{q_1(T)+1} T$$
$$> \frac{\pi^{2/3}(1+m)^{-1/3} m}{2(1+\pi^{2/3}(1+m)^{-1/3} T^{-2/3})} T^{1/3}.$$

Hence

$$\mathcal{A}(\boldsymbol{q}_{\mathrm{col}}) > \frac{3}{2}\pi^{2/3} T^{1/3} + \frac{\pi^{2/3}(1+m)^{-1/3} m}{2(1+\pi^{2/3}(1+m)^{-1/3} T^{-2/3})} T^{1/3}.$$

Under what condition $\boldsymbol{q}^*$ has no collision

- Take a test path : $\boldsymbol{q}_{\text{test}}(t) = (b, ct^{2/3}) \quad (c \geq 0)$
- The action functional of $\boldsymbol{q}_{\text{test}}$:

$$\mathcal{A}(\boldsymbol{q}_{\text{test}}) = \frac{2}{3}c^2 T^{1/3} + \int_0^T \frac{1}{\sqrt{(1-b)^2 + c^2 t^{4/3}}} + \frac{m}{\sqrt{(1+b)^2 + c^2 t^{4/3}}} dt$$

- $F(m, T, c)$

$$:= \tfrac{3}{2}\pi^{2/3} T^{1/3} + \frac{\pi^{2/3}(1+m)^{-1/3} m}{2(1+\pi^{2/3}(1+m)^{-1/3} T^{-2/3})} T^{1/3} - \mathcal{A}(\boldsymbol{q}_{\text{test}})$$

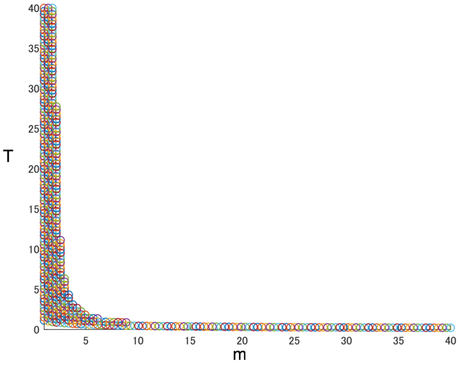
$F(m, T, c) \geq 0 \Rightarrow \mathcal{A}(\boldsymbol{q}_{\text{col}}) > \mathcal{A}(\boldsymbol{q}_{\text{test}}) \geq \mathcal{A}(\boldsymbol{q}^*)$

$\rightarrow F(m, T, c) \geq 0 \Rightarrow \boldsymbol{q}^*$ has no collision.

Main theorem

If $(m, T) \in D$, then there exists a brake orbit which has $4T$-period, where $D := \{(m, T) \mid T > \gamma \,, F(m, T, c) \geq 0 \, ({}^{\exists}c \geq 0)\}$.

Draw the domain D with Matlab.



What kind of brake orbits are minimizers?

Red line : the x-component of the force is zero

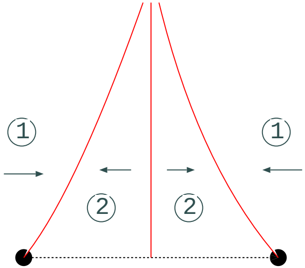


Figure: $m_1 = m_2$

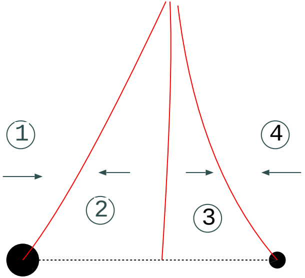


Figure: $m_1 < m_2$

Thank you for your attention!