Variational proof of the existence of brake orbits
in the planar 2-center problem

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$n$-center problem & brake orbits

- $n$-center problem:

$$\ddot{q} = -\sum_{k=1}^{n} \frac{m_k}{|q - a_k|^3} (q - a_k) \quad (q \in \mathbb{R}^d)$$

where $a_1, \ldots, a_n \in \mathbb{R}^d$ are constant vectors.

- $q(t)$ is a brake orbit.

$$\Leftrightarrow \exists T_2 > \exists T_1 > 0 \text{ such that } \dot{q}(T_1) = \dot{q}(T_2) = 0.$$ (Excluding equilibrium points)

- In the potential systems, brake orbits are $2(T_2 - T_1)$-periodic solutions.
Q. Do brake orbits exist in the planar 2-center problem?

Setting: \( m_1 = 1, \ m_2 = m, \ a_1 = a = (1, 0), \ a_2 = -a = (-1, 0). \)

→ Variational methods

Lagrangian: \( L(q, \dot{q}) = \frac{1}{2} |\dot{q}|^2 + \frac{1}{|q - a|} + \frac{m}{|q + a|} \)

Action functional: \( \mathcal{A}(q) = \int_0^T L(q, \dot{q}) dt \)

Boundary conditions: \( q(0) \in A := \{(x, 0) \mid -1 \leq x \leq 1\}, \ q(T) \in \mathbb{R}^2. \)

\[ \mathcal{A}'(q) = 0 \iff q \text{ is a solution of the planar 2-center problem.} \]
Existence of minimizer

Facts:

- With standard arguments of the existence of a minimizer, there is a minimizer \( q^* \) of \( A(q) \) satisfying the boundary condition.
- If \( q^* \) has no collision, \( \dot{q}^*(0) \perp A, \dot{q}^*(T) = 0 \) and \( A'(q^*) = 0 \) hold.
- If:
  
  (Col) \( q^* \) is not a collision solution.
  
  (Eq) \( q^* \) is not an equilibrium point.

Then we obtain a \( 4T \)-periodic brake orbit.
Shape of $q^*$

If $q^*$ satisfies (Col) and (Eq), $q^*$ is a part of a brake orbit, i.e. from $t = 0$ to $t = T$ of $4T$-periodic one.
Shape of the whole brake orbit

$m_2 \quad m_1$
Under what condition $q^*$ is not an equilibrium point

An equilibrium point: $q_{eq} = (b, 0) \left( b = \frac{\sqrt{m} - 1}{\sqrt{m} + 1} \right)$.

(The second variation)

$$A''(q)(\delta) = \lim_{h \to 0} \int (\delta(t), \dot{\delta}(t)) \nabla^2 L |_{(q, \dot{q})=(q+h\delta, \dot{q}+h\dot{\delta})} (\delta(t), \dot{\delta}(t))^T dt$$

→ If $T > \gamma = \frac{\sqrt{2\pi m^{1/4}}}{(1 + \sqrt{m})^2}$, $\exists \delta$ such that $A''(q_{eq})(\delta) < 0$.

→ $T > \gamma \Rightarrow q^*$ is not an equilibrium point.
A minimizer in collision solutions

Let $q_{\text{col}}$ be a minimizer in collision solutions.

We estimate $\mathcal{A}(q_{\text{col}})$:

$$
\mathcal{A}(q_{\text{col}}) = \int_0^T \frac{1}{2} |\dot{q}_{\text{col}}|^2 + \frac{1}{|q_{\text{col}} - a|} \, dt + \int_0^T \frac{m}{|q_{\text{col}} + a|} \, dt
$$

$$
\geq \frac{3}{2} \pi^{2/3} T^{1/3} + \int_0^T \frac{m}{|q_{\text{col}} + a|} \, dt \quad (1)
$$
Estimate the action of collisions

We estimate the second term in (1):

\[ \int_0^T \frac{m}{|q_{\text{col}} + a|} dt = \int_0^T \frac{m}{q_1(t) + 1} dt > \frac{m}{q_1(T) + 1} T \]

\[ > \frac{\pi^{2/3}(1 + m)^{-1/3}m}{2(1 + \pi^{2/3}(1 + m)^{-1/3}T^{-2/3})} T^{1/3}. \]

Hence

\[ \mathcal{A}(q_{\text{col}}) > \frac{3}{2} \pi^{2/3} T^{1/3} + \frac{\pi^{2/3}(1 + m)^{-1/3}m}{2(1 + \pi^{2/3}(1 + m)^{-1/3}T^{-2/3})} T^{1/3}. \]
Under what condition \( q^* \) has no collision

- Take a test path: \( q_{\text{test}}(t) = (b, ct^{2/3}) \quad (c \geq 0) \)

- The action functional of \( q_{\text{test}} \):
  \[
  \mathcal{A}(q_{\text{test}}) = \frac{2}{3}c^2T^{1/3} + \int_0^T \frac{1}{\sqrt{(1-b)^2 + c^2t^{4/3}}} dt + \frac{m}{\sqrt{(1+b)^2 + c^2t^{4/3}}} dt
  \]

- \( F(m, T, c) \)
  \[
  F(m, T, c) := \frac{3}{2} \pi^{2/3}T^{1/3} + \frac{\pi^{2/3}(1+m)^{-1/3}m}{2(1+\pi^{2/3}(1+m)^{-1/3}T^{-2/3})} T^{1/3} - \mathcal{A}(q_{\text{test}})
  \]

\[
F(m, T, c) \geq 0 \Rightarrow \mathcal{A}(q_{\text{col}}) > \mathcal{A}(q_{\text{test}}) \geq \mathcal{A}(q^*)
\]

\[
\rightarrow F(m, T, c) \geq 0 \Rightarrow q^* \text{ has no collision.}
\]
Main theorem

If \((m, T) \in D\), then there exists a brake orbit which has \(4T\)-period, where
\[
D := \{(m, T) \mid T > \gamma, F(m, T, c) \geq 0 (\exists c \geq 0)\}.
\]

Draw the domain \(D\) with Matlab.
What kind of brake orbits are minimizers?

Red line: the x-component of the force is zero

\[ m_1 = m_2 \]  

\[ m_1 < m_2 \]
Thank you for your attention!