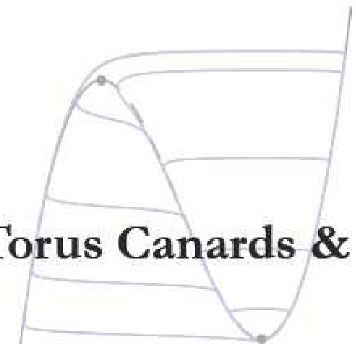
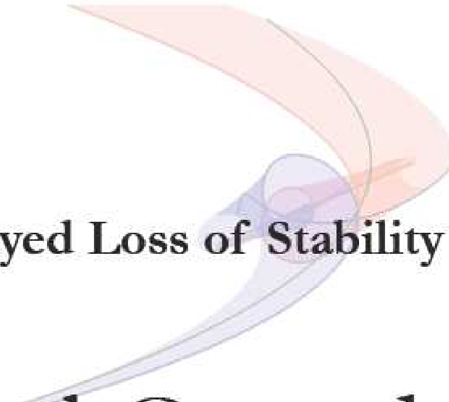
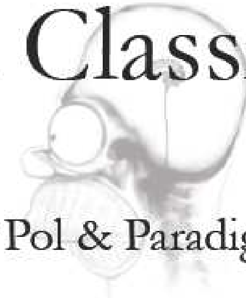
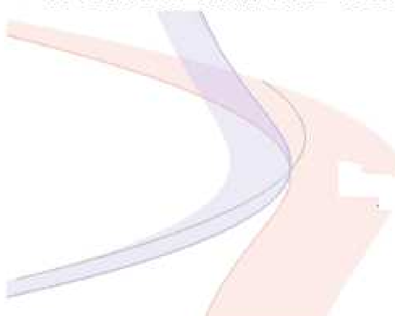
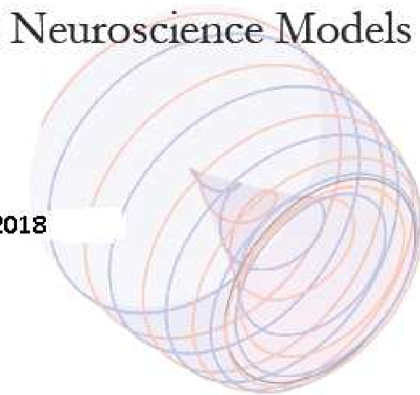



**Analysis of Torus Canards & Other Delayed Loss of Stability Phenomena**

# Toral & Classical Canards



Forced van der Pol & Paradigm Neuroscience Models

June 28, 2018

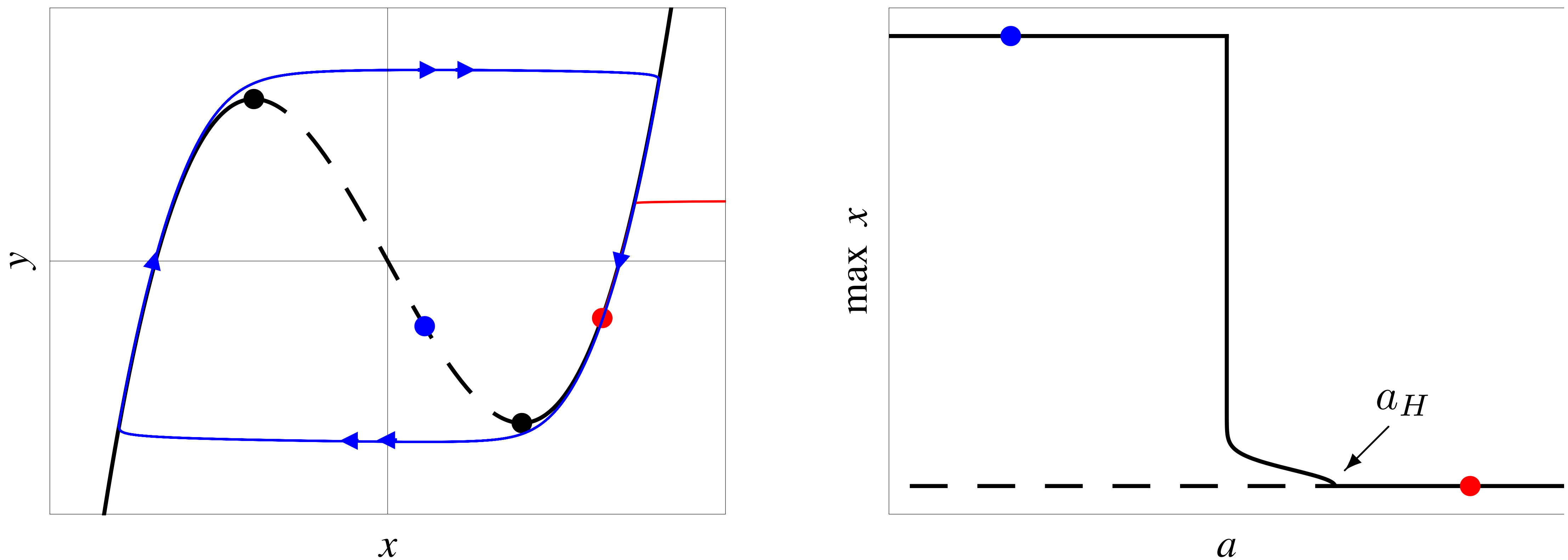
Theodore Vo & Tasso Kaper
Department of Mathematics and Statistics
Boston University

# Outline

- **Review of limit cycle canards**
- **Cast of characters: folded singularity & torus canards**
- **Main results for the forced van der Pol equation**
  - low-frequency forcing
  - intermediate- and high-frequency forcing
- **Impacts of main results**
  - general forced systems
  - transition between periodic spiking and bursting of many different types
  - organization of the SAO, LAO, and MMO
  - generic torus canards
- **Conclusions**

# Canard Cycles in the van der Pol Equation

$$\dot{x} = y - \left(\frac{x^3}{3} - x\right)$$
$$\dot{y} = \varepsilon\left(-x + a\right)$$

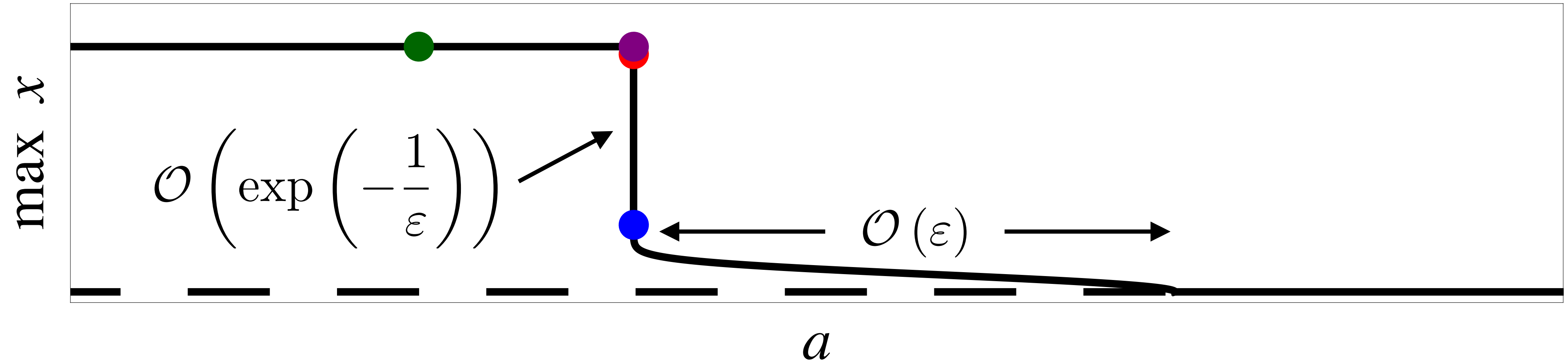


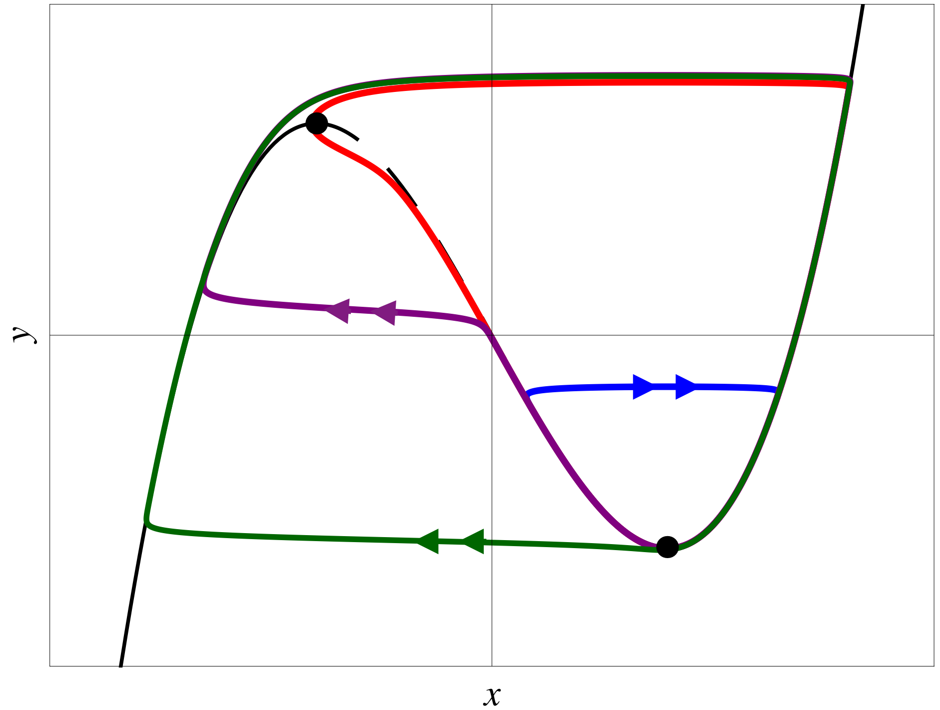

Benoit, Callot, Diener and Diener (1981), Collectanea Mathematicae **31–32**, 37–119
Diener (1984), Math. Intell. **6**, 38–49

- Hopf Bifurcation at $a = 1$
- Canard Explosion at $a = 1 + \mathcal{O}(\varepsilon)$

# Canard Cycles in the van der Pol Equation

$$\dot{x} = y - \left(\frac{x^3}{3} - x\right)$$
$$\dot{y} = \varepsilon\left(-x + a\right)$$

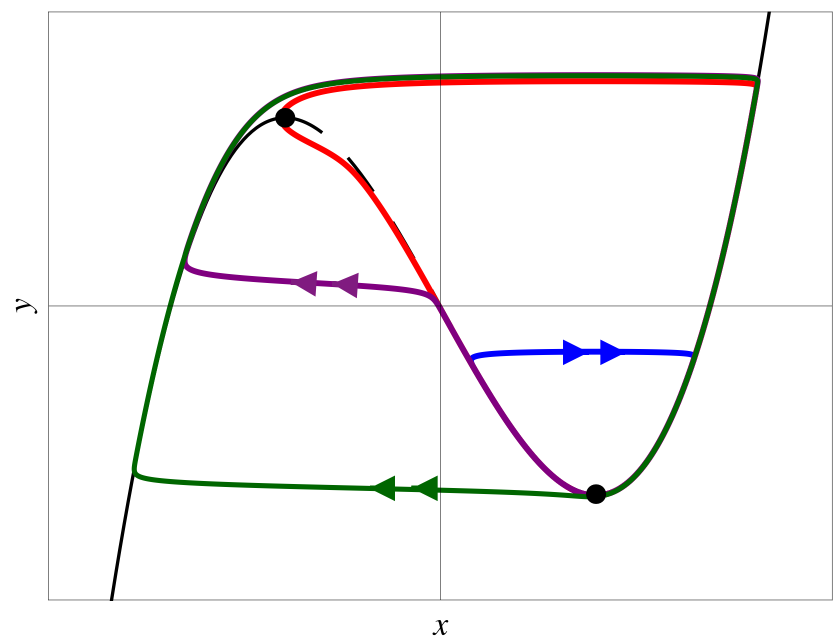


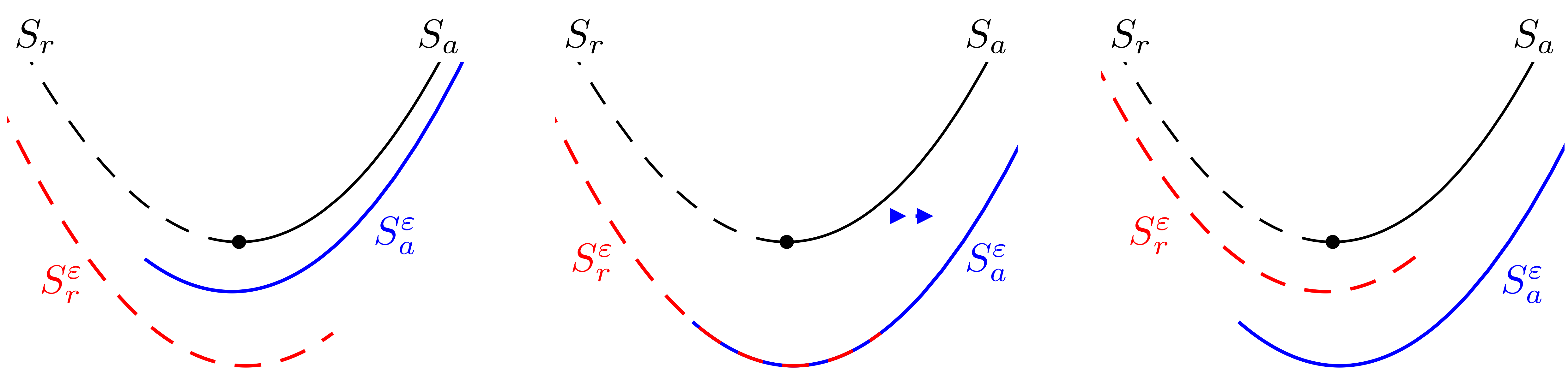




Transition from attracting fixed point to attracting relaxation oscillation must be via sequence of canard cycles

# Canard Cycles in the van der Pol Equation

$$\dot{x} = y - \left(\frac{x^3}{3} - x\right)$$
$$\dot{y} = \varepsilon\left(-x + a\right)$$



# Outline

- **Review of limit cycle canards**
- **Cast of characters: folded singularity & torus canards**
- **Main results for the forced van der Pol equation**
  - low-frequency forcing
  - intermediate- and high-frequency forcing
- **Impacts of main results**
  - general forced systems
  - transition between periodic spiking and bursting of many different types
  - organization of the SAO, LAO, and MMO
  - generic torus canards
- **Conclusions**

# Canards

Fast System

$$x' = f(x, y, \varepsilon)$$
$$y' = \varepsilon\, g(x, y, \varepsilon)$$

$$\xleftrightarrow{t \,=\, \varepsilon\, \tau}$$

Slow System

$$\varepsilon\, \dot{x} = f(x, y, \varepsilon)$$
$$\dot{y} = g(x, y, \varepsilon)$$

$$\Big\downarrow \; \varepsilon \to 0$$

Fast Subsystem

$$x' = f(x, y, 0)$$
$$y' = 0$$

$$\Big\downarrow \; \varepsilon \to 0$$

Slow Subsystem

$$0 = f(x, y, 0)$$
$$\dot{y} = g(x, y, 0)$$

$$0 < \varepsilon \ll 1$$

$x \in \mathbb{R}^m$ ... Fast

$y \in \mathbb{R}^n$ ... Slow

$$\boxed{S := \{\ (x, y)\ :\ f(x, y, 0)\ =\ 0\ \}}$$

**Canard:** solution that spends $\mathcal{O}(1)$ slow time on a repelling manifold of states (equilibria/limit cycles/$n$-torus) of the fast subsystem

# Canards

**Fast System**

$$x' = f(x, y, \varepsilon)$$
$$y' = \varepsilon\, g(x, y, \varepsilon)$$

$$\xleftrightarrow{\;t \,=\, \varepsilon\, \tau\;}$$

**Slow System**

$$\varepsilon\, \dot{x} = f(x, y, \varepsilon)$$
$$\dot{y} = g(x, y, \varepsilon)$$

$$\varepsilon \to 0$$

**Fast Subsystem**

$$x' = f(x, y, 0)$$
$$y' = 0$$

$$\varepsilon \to 0$$

**Slow Subsystem**

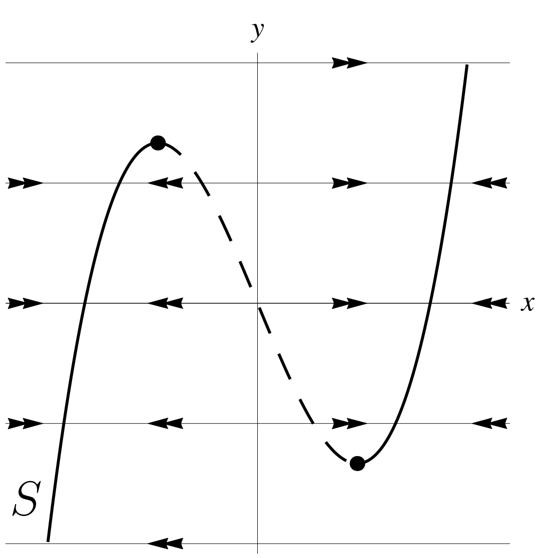
$$0 = f(x, y, 0)$$
$$\dot{y} = g(x, y, 0)$$

$$0 < \varepsilon \ll 1$$

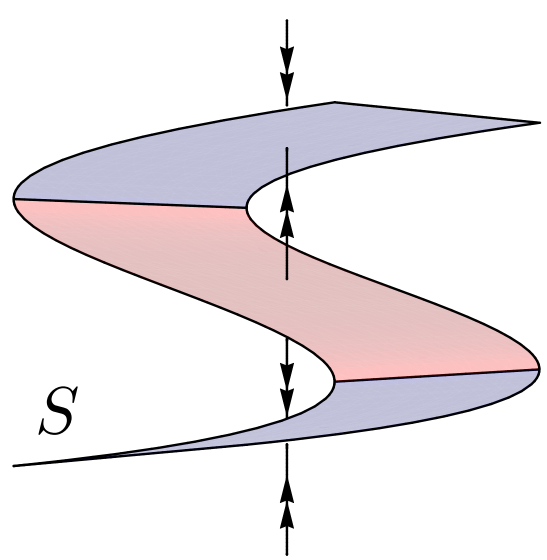
$x \in \mathbb{R}^m$ . . . Fast

$y \in \mathbb{R}^n$ . . . Slow

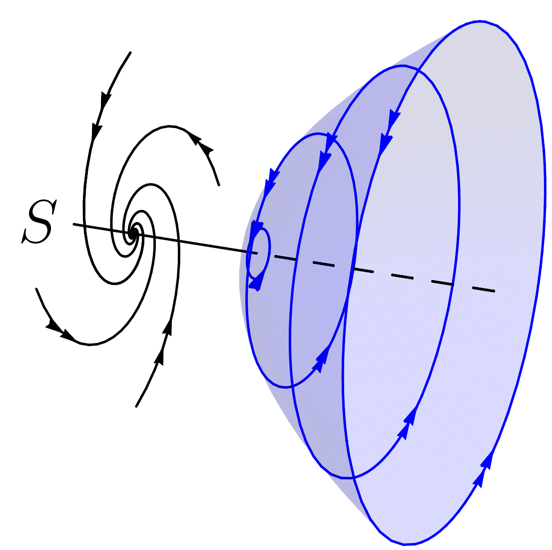
$$\boxed{S := \{\ (x, y)\ :\ f(x, y, 0)\ =\ 0\ \}}$$



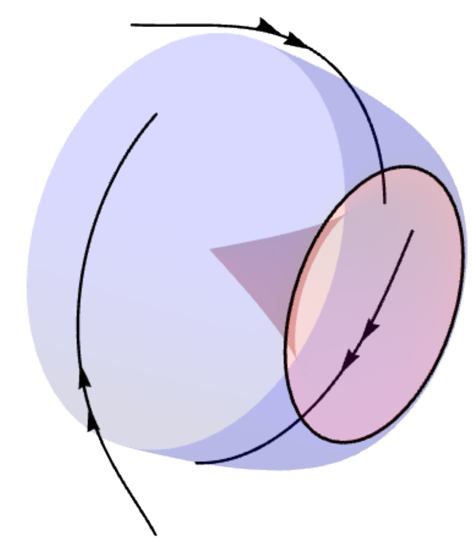

1D + Fold



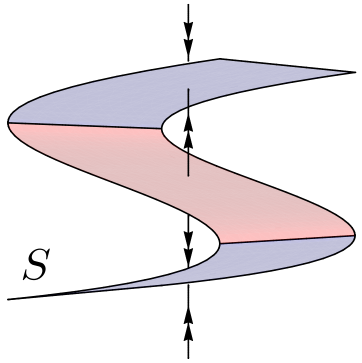

2D + Fold



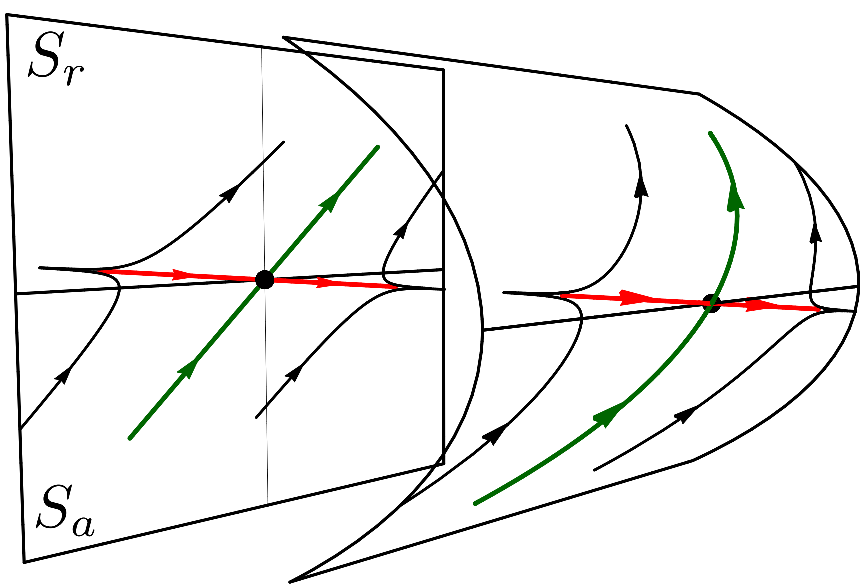

1D + Hopf

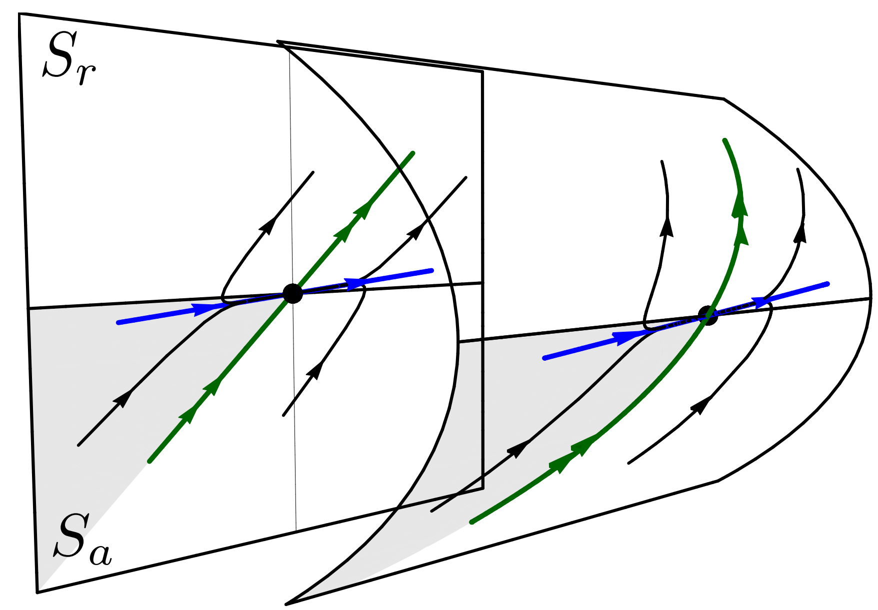


2D + SNPO

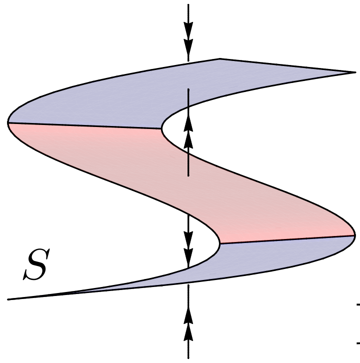



# Folded Singularity Canards[1,2]

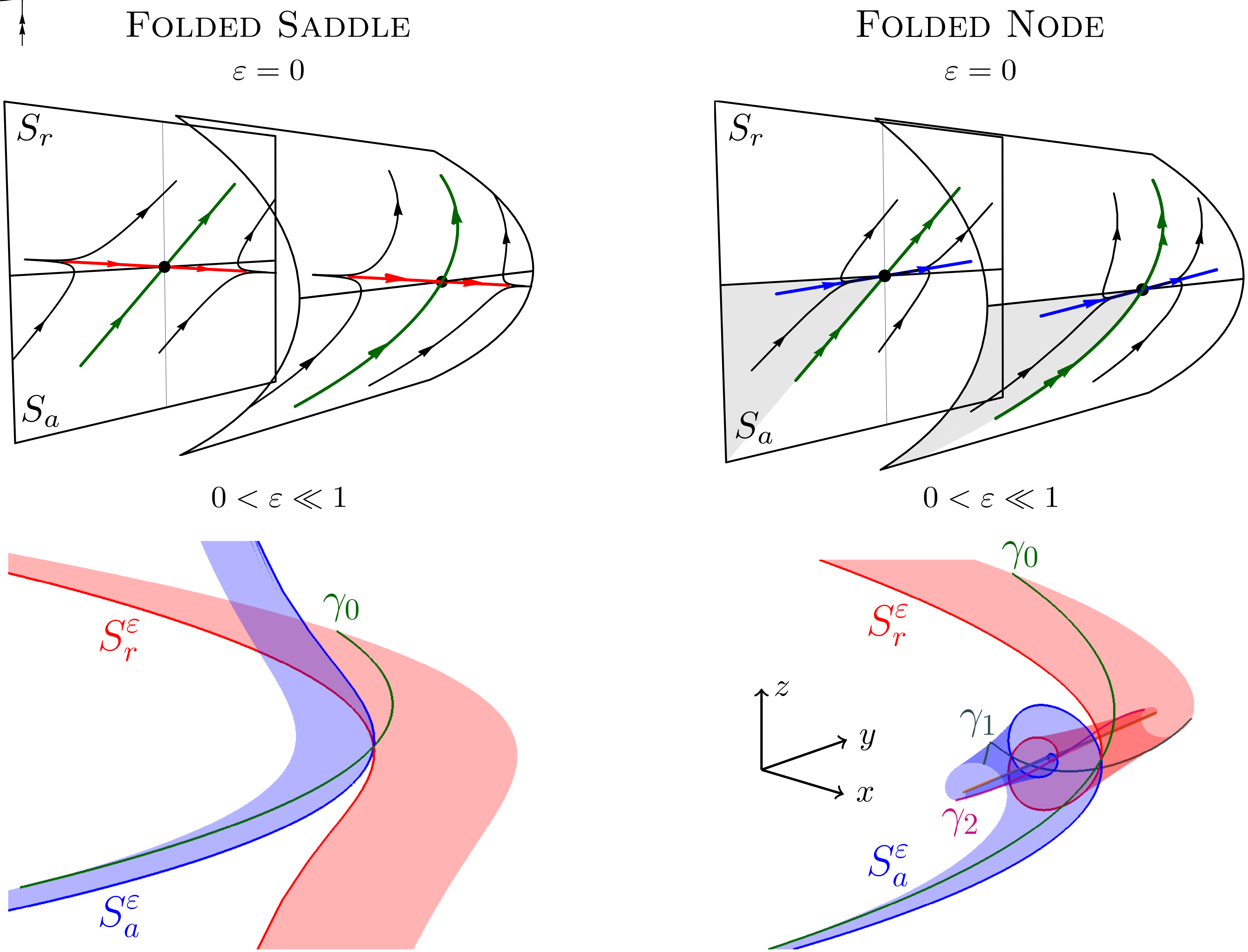


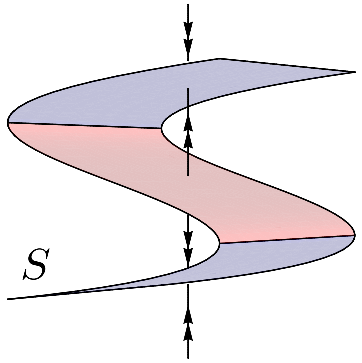

1. Szmolyan & Wechselberger (2001) J. Differential Equ. **177**, 419-453
2. Wechselberger (2005) SIAM J. Appl. Dyn. Syst. **4**, 101-139



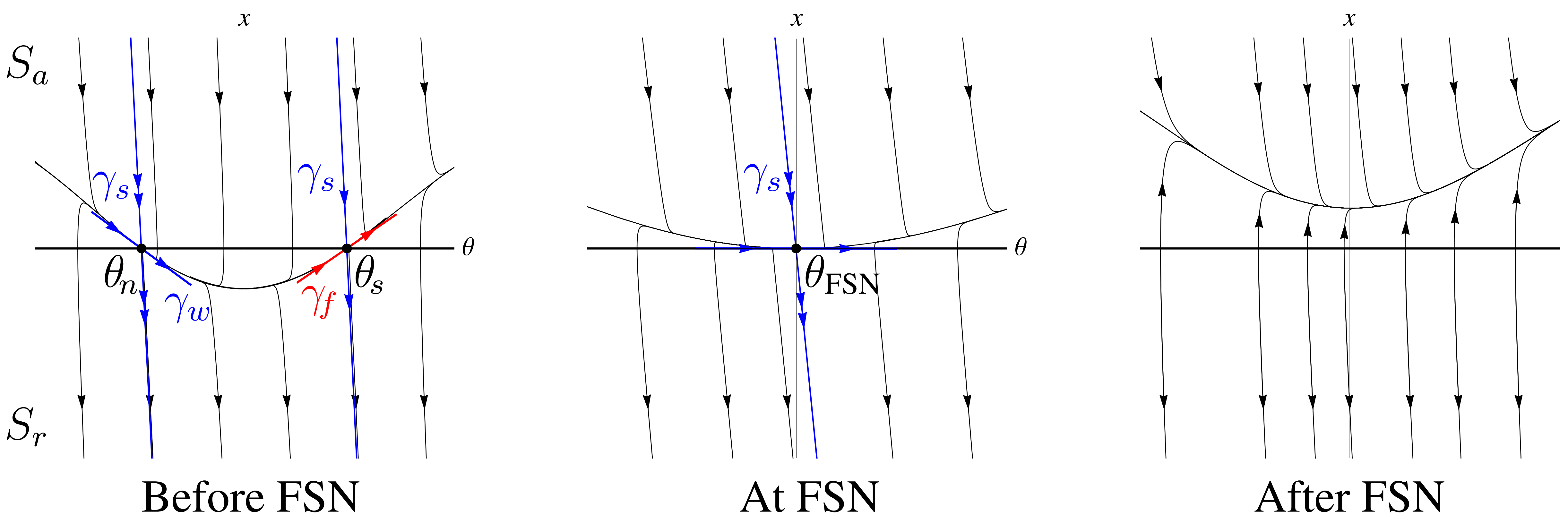

# Folded Singularity Canards[1,2]



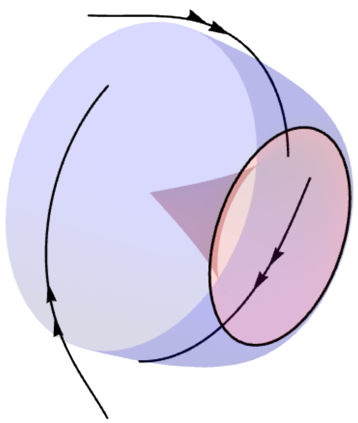

1. Szmolyan & Wechselberger (2001) J. Differential Equ. **177**, 419-453
2. Wechselberger (2005) SIAM J. Appl. Dyn. Syst. **4**, 101-139




# Folded Saddle-Node Canards

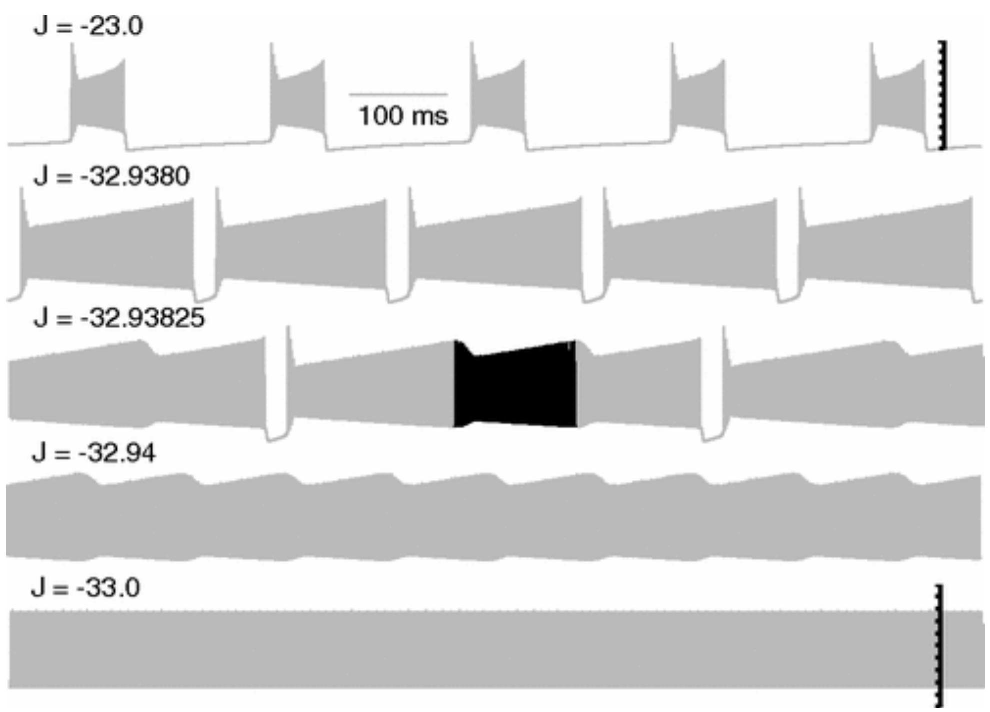


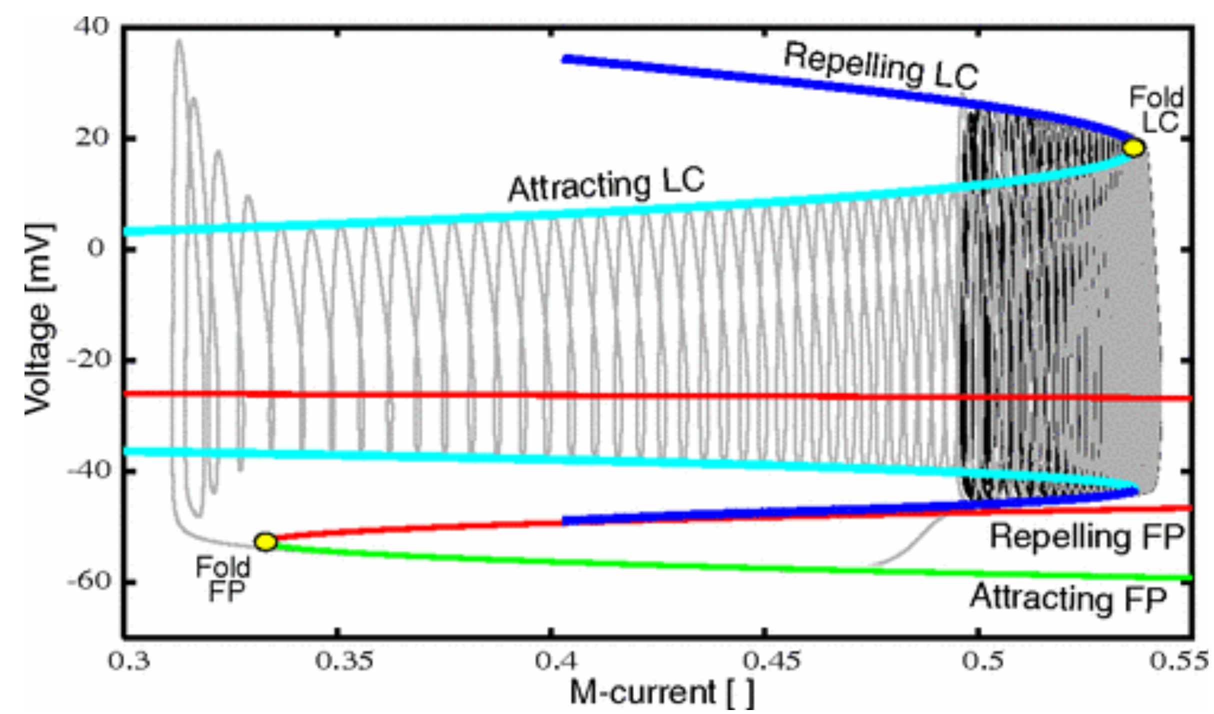

Vo & Wechselberger (2015) SIAM J. Math. Anal. **47**, 3235-3283



# Torus Canards

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2, y) & \qquad x_1 \;\ldots\; \textsc{Fast} \\ \dot{x}_2 &= f_2(x_1, x_2, y) & \qquad x_2 \;\ldots\; \textsc{Fast} \\ \dot{y} &= \varepsilon\, g(x_1, x_2, y) & \qquad y \;\ldots\; \textsc{Slow} \end{aligned}$$




**Discovery:** Kramer, Traub & Kopell (2008), Phys. Rev. Lett. **101**, 068103

- Torus bifurcation for $0 < \varepsilon \ll 1$
- $\varepsilon = 0$ subsystem has fold of limit cycles (aka SNPO)

# Outline

- **Review of limit cycle canards**
- **Cast of characters: folded singularity & torus canards**
- **Main results for the forced van der Pol equation**
  - low-frequency forcing
  - intermediate- and high-frequency forcing
- **Impacts of main results**
  - general forced systems
  - transition between periodic spiking and bursting of many different types
  - organization of the SAO, LAO, and MMO
  - generic torus canards
- **Conclusions**

# The Forced van der Pol Equation[1-4]

$$\dot{x} = y - \left(\frac{x^3}{3} - x\right)$$
$$\dot{y} = \varepsilon\left(-x + a + b\cos\theta\right)$$
$$\dot{\theta} = \omega$$

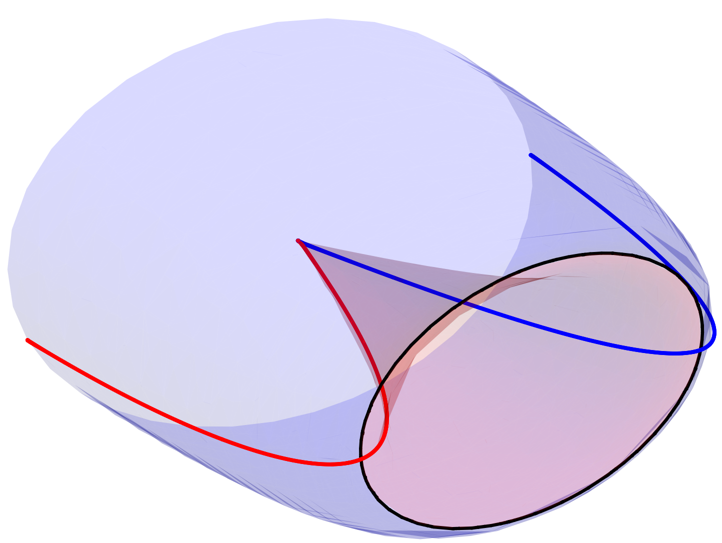

1. van der Pol (1920), Radio Rev. **1**, 701–710, 754–762
2. van der Pol (1927), Lond. Edinb. Dublin Phil. Mag. J. Sci. Seri. 7, **3**, 65-80
3. Cartwright & Littlewood (1945), J. Lond. Math. Soc. **20**, 180-189
4. Burke, Desroches, Granados, Kaper, Krupa & Vo (2016), J. Nonlinear Sci. **26**, 405-451


## Other types of forcing of the van der Pol equation[5,6]

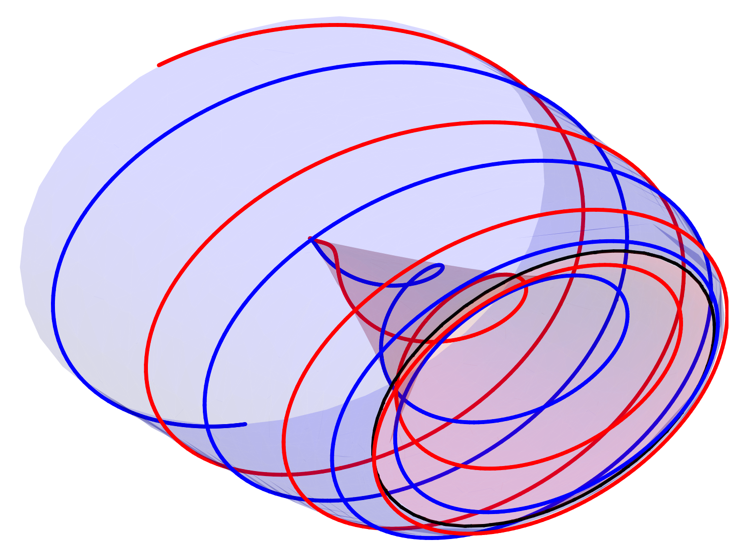

5. Guckenheimer, Hoffman & Weckesser (2003), SIAM J. Appl. Dyn. Syst. **2**, 1-35
6. Haiduc (2009) Nonlinearity **22**, 213-237

# The Forced van der Pol Equation

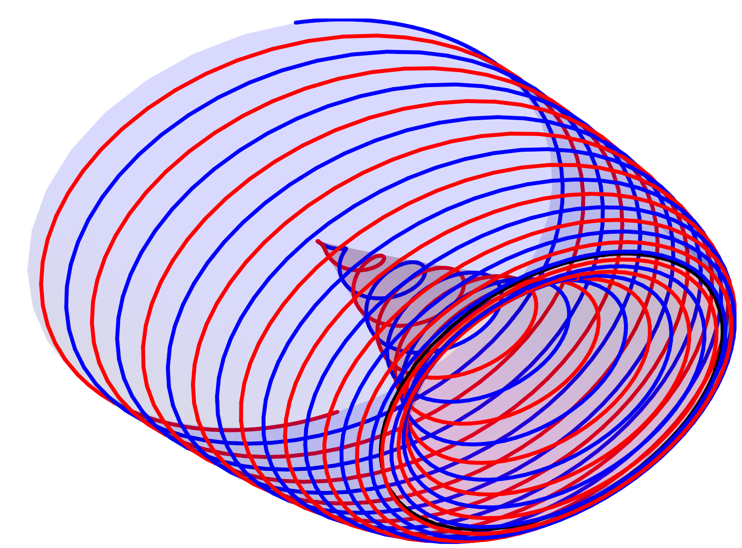
$$\dot{x} = y - \left(\frac{x^3}{3} - x\right)$$
$$\dot{y} = \varepsilon\left(-x + a + b\cos\theta\right)$$
$$\dot{\theta} = \omega$$



$\omega = \mathcal{O}\left(\varepsilon\right)$



$\omega = \mathcal{O}\left(\sqrt{\varepsilon}\right)$



$\omega = \mathcal{O}\left(1\right)$

# Low-Frequency Forcing ($\omega = \varepsilon\,\overline{\omega}$): Fold Curve

Fast System

$$\begin{aligned} x' &= y - \left(\frac{x^3}{3} - x\right) \\ y' &= \varepsilon\,(-x + a + b\,\cos\theta) \\ \theta' &= \varepsilon\,\overline{\omega} \end{aligned}$$

$$\xleftrightarrow{\;t\,=\,\varepsilon\,\tau\;}$$

Slow System

$$\begin{aligned} \varepsilon\,\dot{x} &= y - \left(\frac{x^3}{3} - x\right) \\ \dot{y} &= -x + a + b\,\cos\theta \\ \dot{\theta} &= \overline{\omega} \end{aligned}$$

$$\varepsilon \downarrow 0$$

Fast Subsystem

$$\begin{aligned} x' &= y - \left(\frac{x^3}{3} - x\right) \\ y' &= 0 \\ \theta' &= 0 \end{aligned}$$

Slow Subsystem

$$\begin{aligned} 0 &= y - \left(\frac{x^3}{3} - x\right) \\ \dot{y} &= -x + a + b\,\cos\theta \\ \dot{\theta} &= \overline{\omega} \end{aligned}$$

$$\boxed{S := \left\{ (x, y, \theta) \,:\, y = \frac{x^3}{3} - x \right\}}$$

# Low-Frequency Forcing ($\omega = \varepsilon\,\overline{\omega}$): Folded Singularities

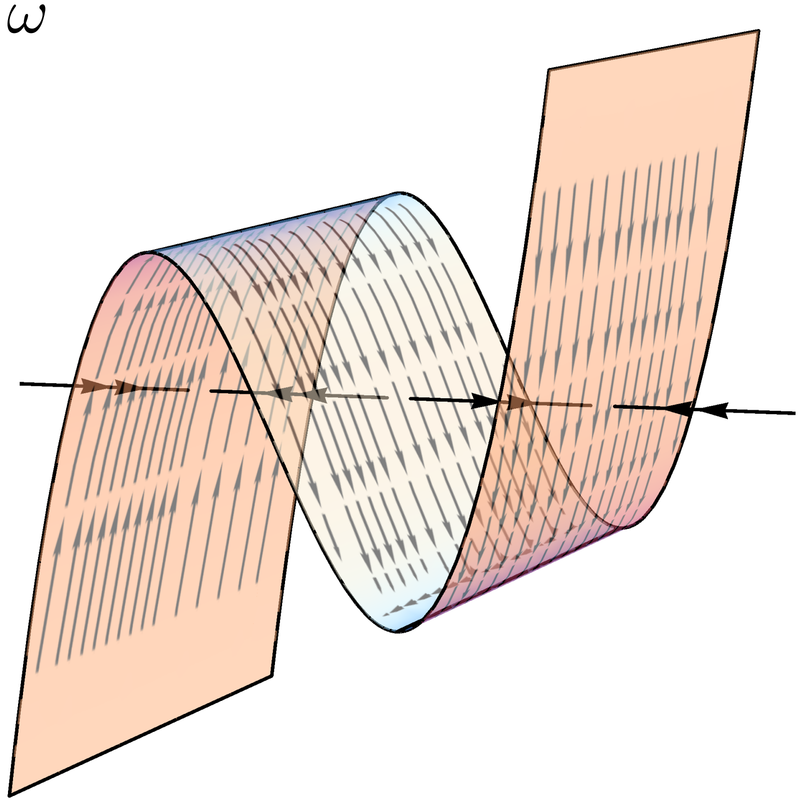
$$0 = y - \left(\frac{x^3}{3} - x\right)$$
$$\dot{y} = -x + a + b\cos\theta$$
$$\dot{\theta} = \overline{\omega}$$

$$\xrightarrow{\frac{d}{dt}0 = \frac{d}{dt}\left(y - (\frac{x^3}{3} - x)\right)}$$

Projection Onto $(x, \theta)$

$$(x^2 - 1)\,\dot{x} = -x + a + b\cos\theta$$
$$\dot{\theta} = \overline{\omega}$$

$$\Big\downarrow \; dt = (x^2 - 1)\,ds$$



Desingularized System

$$\dot{x} = -x + a + b\cos\theta$$
$$\dot{\theta} = \overline{\omega}\,(x^2 - 1)$$

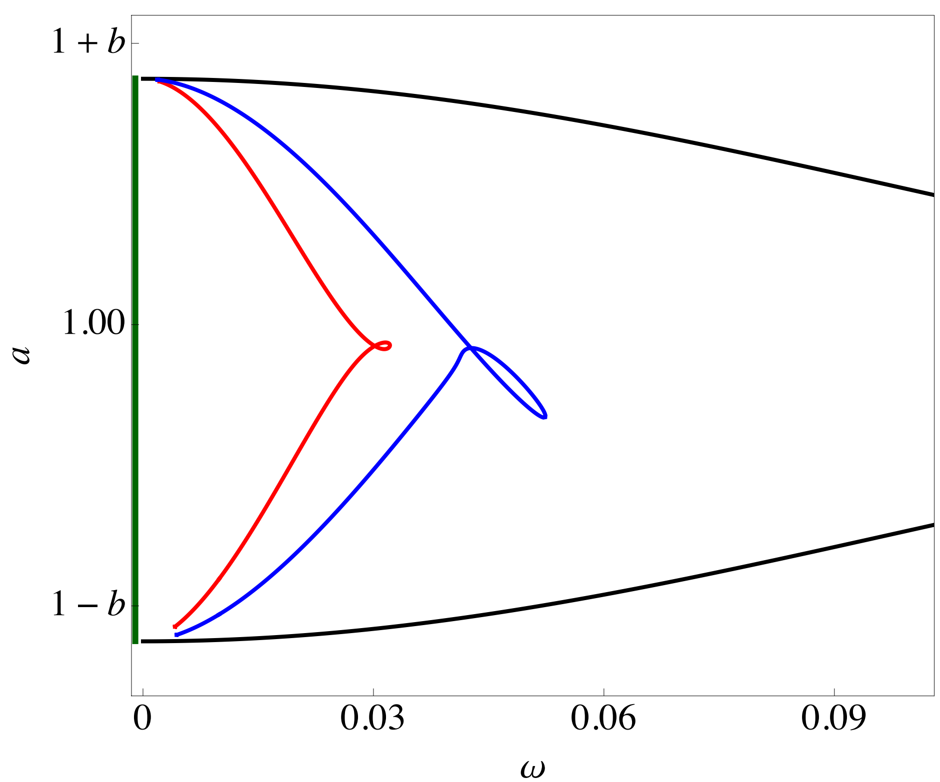
$$\boxed{\text{Folded Singularities}: \quad (x, y, \theta) = \left(1, -\frac{2}{3}, \pm\cos^{-1}\left(\frac{1-a}{b}\right)\right)}$$

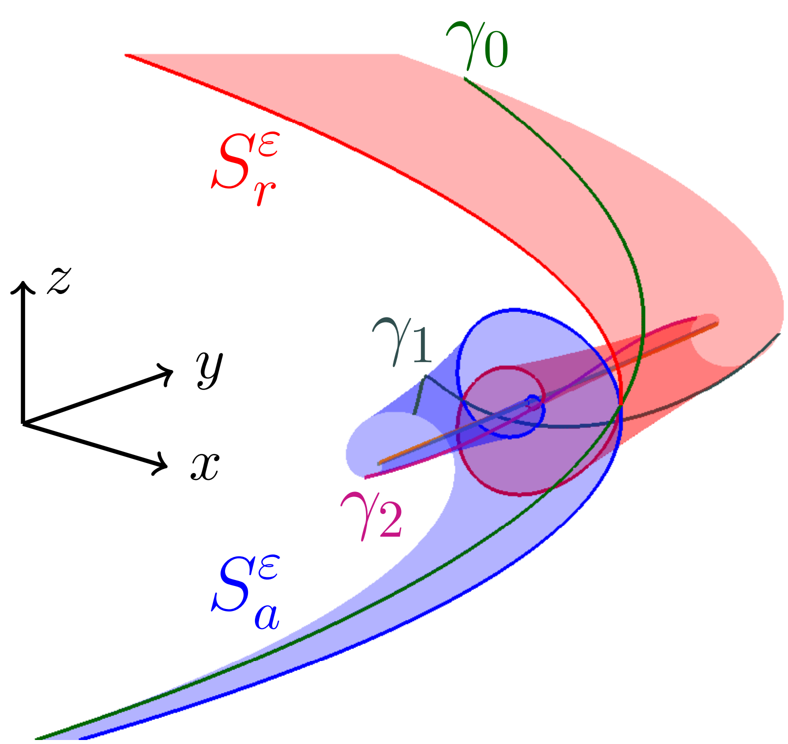
$\theta_S = \cos^{-1}\left(\frac{1-a}{b}\right)$ … Folded Node

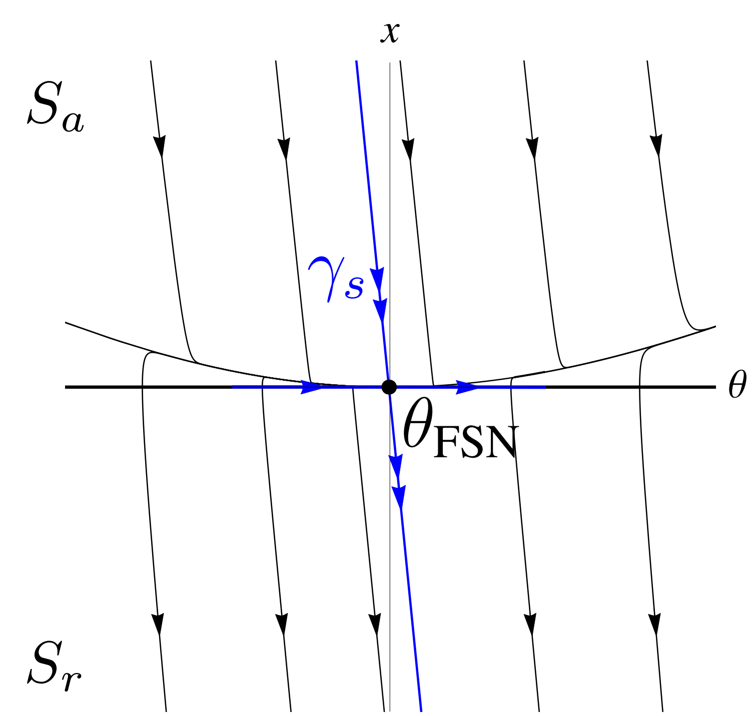
$\theta_S = -\cos^{-1}\left(\frac{1-a}{b}\right)$ … Folded Saddle

# Low-Frequency Forcing ( $\omega = \varepsilon \overline{\omega}$ ): Folded Singularity Canards

Folded Saddle-Node (Type I) Bifurcation at $a = 1 \pm b$ for $\varepsilon = 0$




At FSN

Idea: geometric desingularization of the FSN point

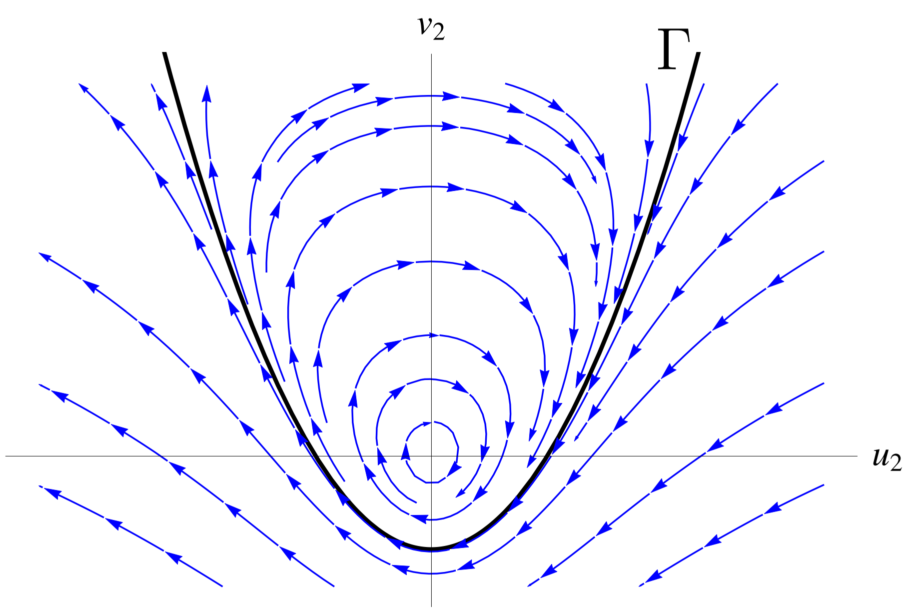
# Low-Frequency Forcing ( $\omega = \varepsilon\,\overline{\omega}$ ): Folded Singularity Canards

Geometric desingularization of the FSN I

Blow-up : $x = 1 + \sqrt{\varepsilon}\,u_2 \quad y = -\frac{2}{3} + \varepsilon\,v_2 \quad \theta = \varepsilon^{1/4}\,\theta_2 \quad a = 1 \pm b + \varepsilon\,\alpha_2 \quad b = \sqrt{\varepsilon}\,\beta_2 \quad t = \frac{1}{\sqrt{\varepsilon}}\,t_2$

$$\dot{u}_2 = v_2 - u_2^2 - \frac{1}{3}\sqrt{\varepsilon}\,u_2^3$$

$$\dot{v}_2 = -u_2 + \sqrt{\varepsilon}\,\alpha_2 + \beta_2 \underbrace{\left(\cos(\sqrt{\varepsilon}\overline{\omega}t_2)\cos(\varepsilon^{1/4}\theta_{2,0}) - 1\right)}_{\mathcal{O}(\sqrt{\varepsilon})} - \beta_2 \underbrace{\sin(\sqrt{\varepsilon}\overline{\omega}t_2)\,\sin(\varepsilon^{1/4}\theta_{2,0})}_{\mathcal{O}(\varepsilon^{3/4})}$$



# Low-Frequency Forcing ( $\omega = \varepsilon\,\overline{\omega}$ ): Folded Singularity Canards

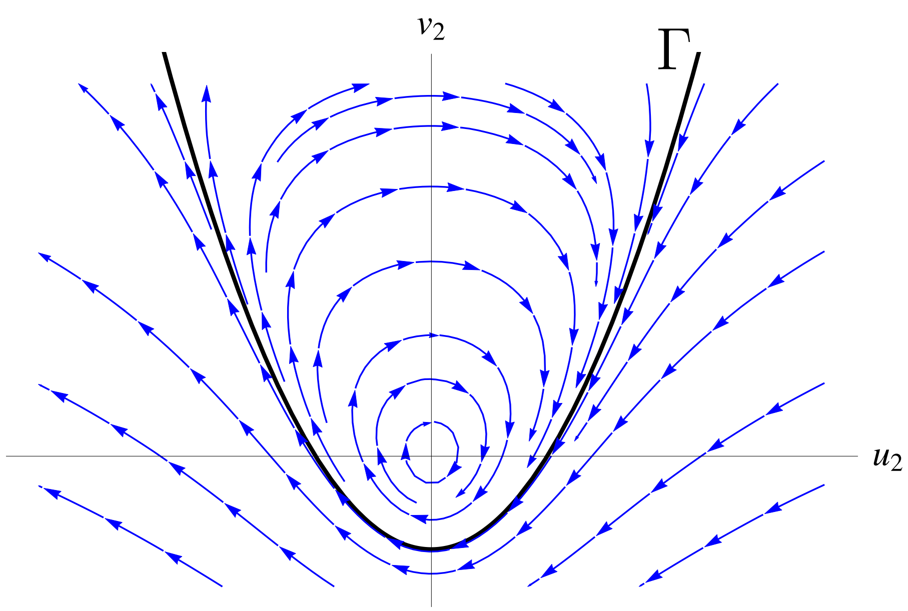
Geometric desingularization of the FSN I

Blow-up : $x = 1+\sqrt{\varepsilon}\,u_2 \quad y = -\frac{2}{3}+\varepsilon\,v_2 \quad \theta = \varepsilon^{1/4}\,\theta_2 \quad a = 1\pm b+\varepsilon\,\alpha_2 \quad b = \sqrt{\varepsilon}\,\beta_2 \quad t = \frac{1}{\sqrt{\varepsilon}}\,t_2$

$$\dot{u}_2 = v_2 - u_2^2 - \frac{1}{3}\sqrt{\varepsilon}\,u_2^3$$
$$\dot{v}_2 = -u_2 + \sqrt{\varepsilon}\,\alpha_2 + \beta_2 \underbrace{\left(\cos(\sqrt{\varepsilon}\overline{\omega}t_2)\cos(\varepsilon^{1/4}\theta_{2,0}) - 1\right)}_{\mathcal{O}(\sqrt{\varepsilon})} - \beta_2 \underbrace{\sin(\sqrt{\varepsilon}\overline{\omega}t_2)\,\sin(\varepsilon^{1/4}\theta_{2,0})}_{\mathcal{O}(\varepsilon^{3/4})}$$

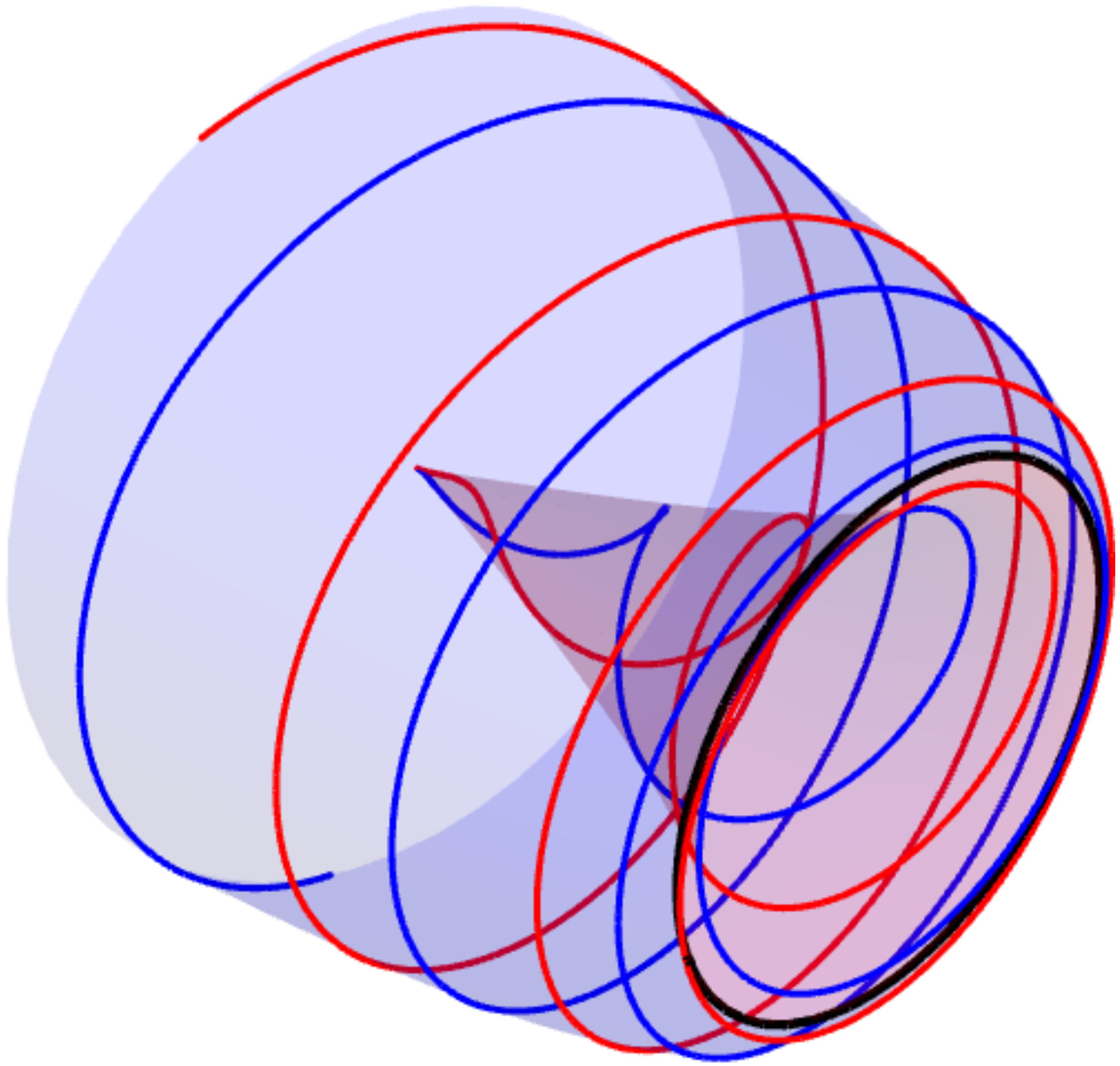
$$\mathcal{D} = d_1\,\sqrt{\varepsilon} + d_2\,\varepsilon^{3/4} + \cdots = 0$$

$$d_1 = \int_{-\infty}^{\infty} \nabla H|_\Gamma \cdot \begin{pmatrix} -\frac{1}{3}u_\Gamma^2 \\ \alpha_2 + \frac{\beta_2}{\sqrt{\varepsilon}}\left(\cos(\sqrt{\varepsilon}\overline{\omega}t_2)\cos(\varepsilon^{1/4}\theta_{2,0}) - 1\right) \end{pmatrix} dt_2$$
$$d_2 = \int_{-\infty}^{\infty} \nabla H|_\Gamma \cdot \begin{pmatrix} 0 \\ -\frac{\beta_2}{\varepsilon^{3/4}}\sin(\sqrt{\varepsilon}\overline{\omega}t_2)\sin(\varepsilon^{1/4}\theta_{2,0}) \end{pmatrix} dt_2$$




$$\boxed{a = 1 - \frac{\varepsilon}{8} \pm b\,\exp\left(-\frac{\omega^2}{2\varepsilon}\right)}$$

# Torus Canards: Intermediate-Frequency Forcing

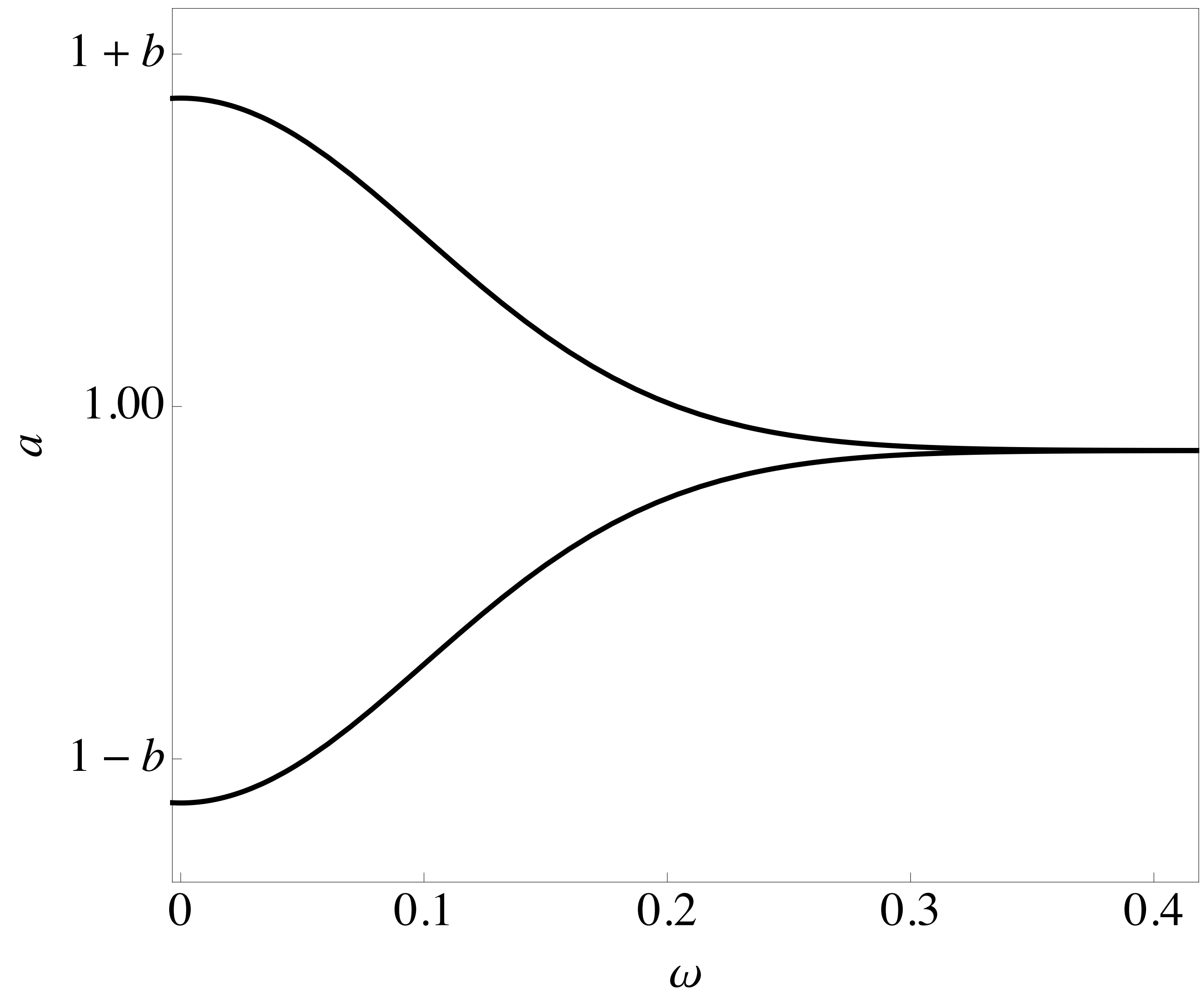


Idea: geometric desingularization of the SNPO curve

$$\omega = \mathcal{O}\left(\sqrt{\varepsilon}\right)$$

# Main Result[1]

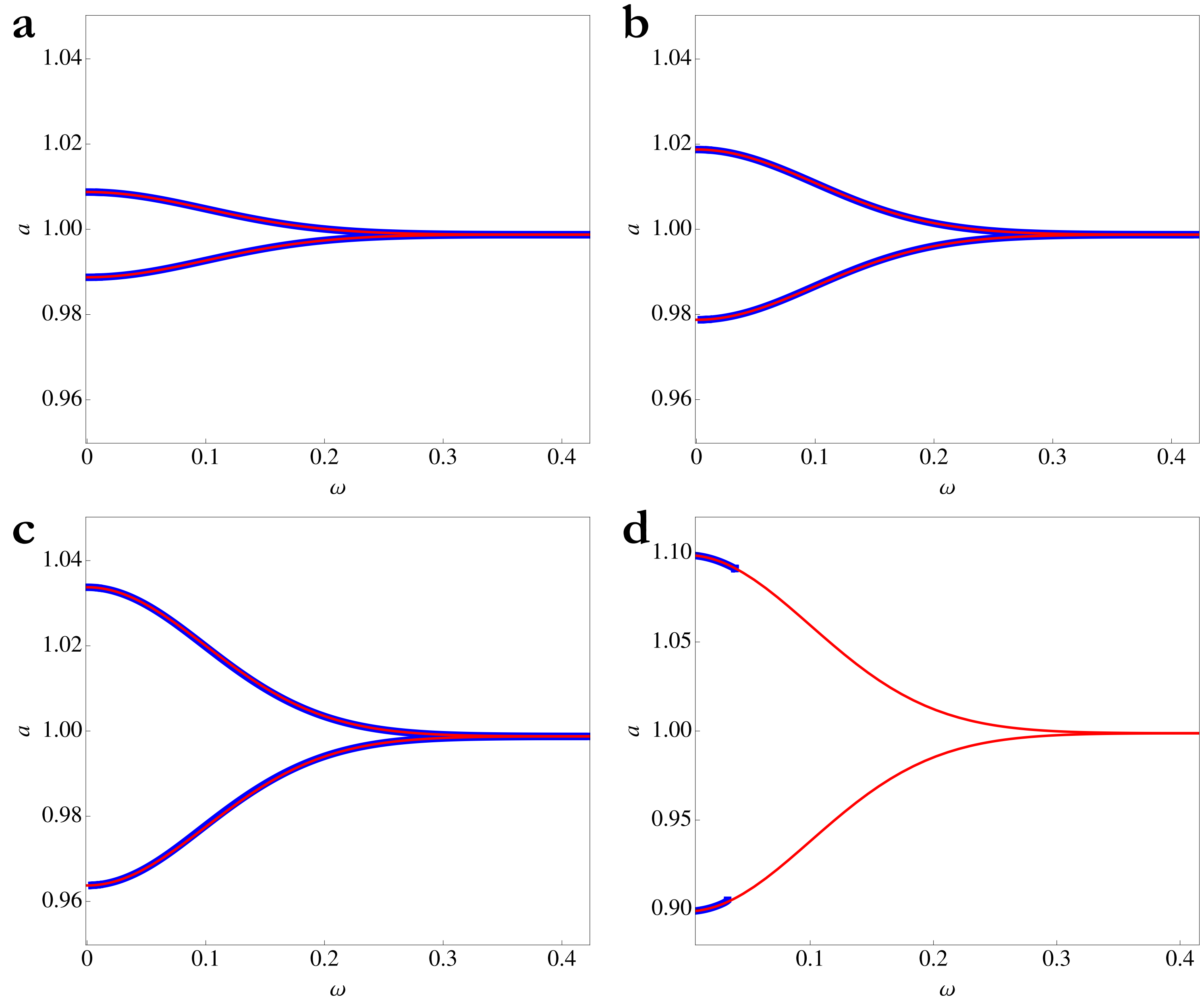
$$\dot{x} = y - \left(\frac{x^3}{3} - x\right)$$
$$\dot{y} = \varepsilon\ (-x + a + b\cos\theta)$$
$$\dot{\theta} = \omega$$




$$\boxed{a = 1 - \frac{\varepsilon}{8} \pm b\exp\left(-\frac{\omega^2}{2\varepsilon}\right)}$$

1. Burke, Desroches, Granados, Kaper, Krupa & Vo (2016), J. Nonlinear Sci. **26**, 405–451

# Main Result[1]




Curves of folds of maximal canards in the $(\omega, a)$ plane as obtained from the analytic formula (red curves) and 2-parameter numerical continuation (blue curves) for $\varepsilon = 0.01$ and (**a**) $b = 0.01$; (**b**) $b = 0.02$; (**c**) $b = 0.035$; and, (**d**) $b = 0.1$. For $b = \mathcal{O}(\varepsilon)$ (**a**-**c**), there is good agreement between theoretical (red) and numerical (blue) results over the entire range of forcing frequencies, including for both the primary maximal canards which exist for $\omega = \mathcal{O}(\varepsilon)$ and the maximal torus canards which exist for $\omega = \mathcal{O}(1)$. Note that the scales in **a**-**c** are the same. For $b = \mathcal{O}(\sqrt{\varepsilon})$ (**d**, in which the vertical scale is different), we find that the numerical continuation terminates when $\omega$ is no longer $\mathcal{O}(\varepsilon)$.

1. Burke, Desroches, Granados, Kaper, Krupa & Vo (2016), J. Nonlinear Sci. **26**, 405–451

# Outline

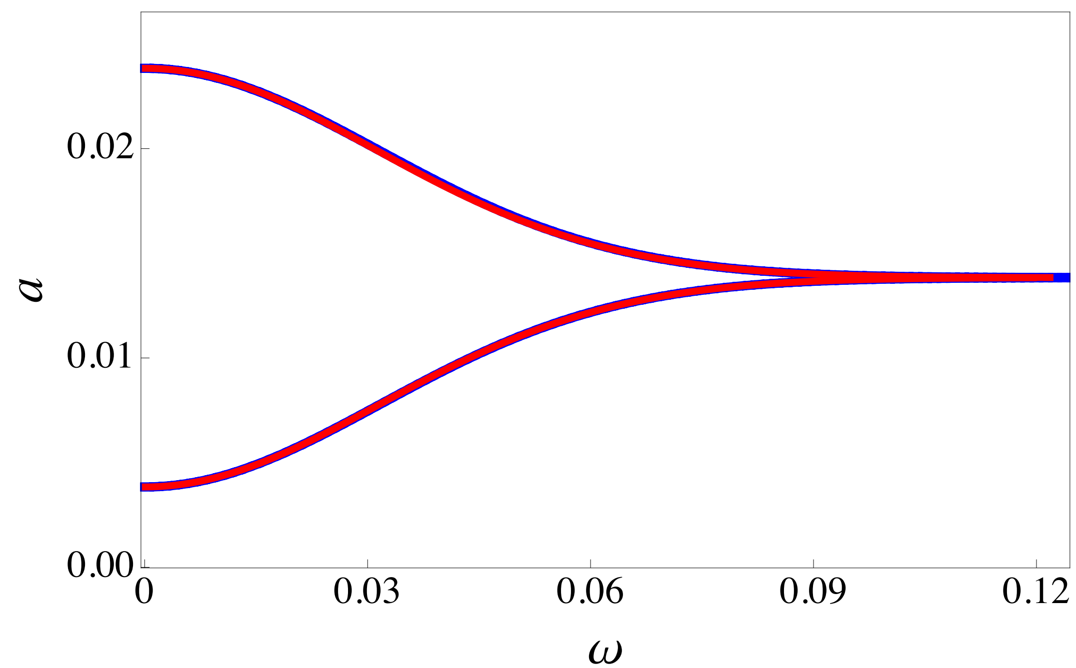
- **Review of limit cycle canards**
- **Cast of characters: folded singularity & torus canards**
- **Main results for the forced van der Pol equation**
  - low-frequency forcing
  - intermediate- and high-frequency forcing
- **Impacts of main results**
  - general forced systems
  - transition between periodic spiking and bursting of many different types
  - organization of the SAO, LAO, and MMO
  - generic torus canards
- **Conclusions**

Why do we care? Reason 1

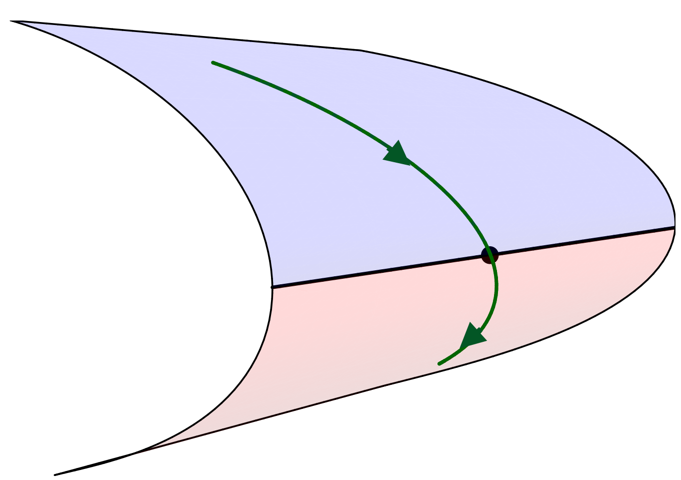
# Toral & Classical Canards in Forced Planar Slow/Fast Systems

Example: forced FitzHugh-Nagumo

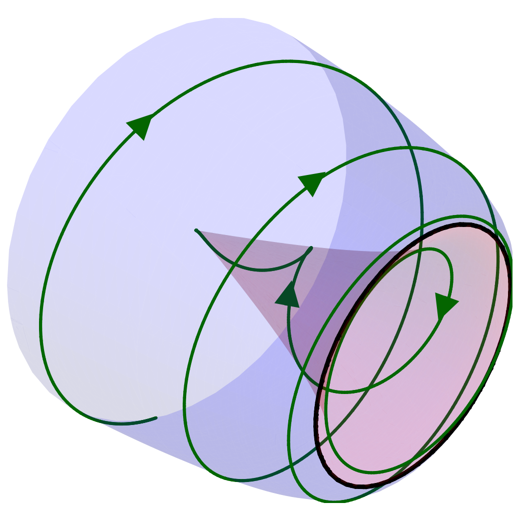
$$\dot{x} = x - \frac{x^3}{3} - y + I$$
$$\dot{y} = \varepsilon\,(x + a - cy + b\cos\theta)$$
$$\dot{\theta} = \omega$$




$$\boxed{a = -1 + \left(\frac{2}{3} + I\right)c + \frac{1+2c}{8}\varepsilon \pm b\,\exp\left(-\frac{\omega^2}{2\varepsilon}\right)}$$



$\omega = \mathcal{O}(\varepsilon)$



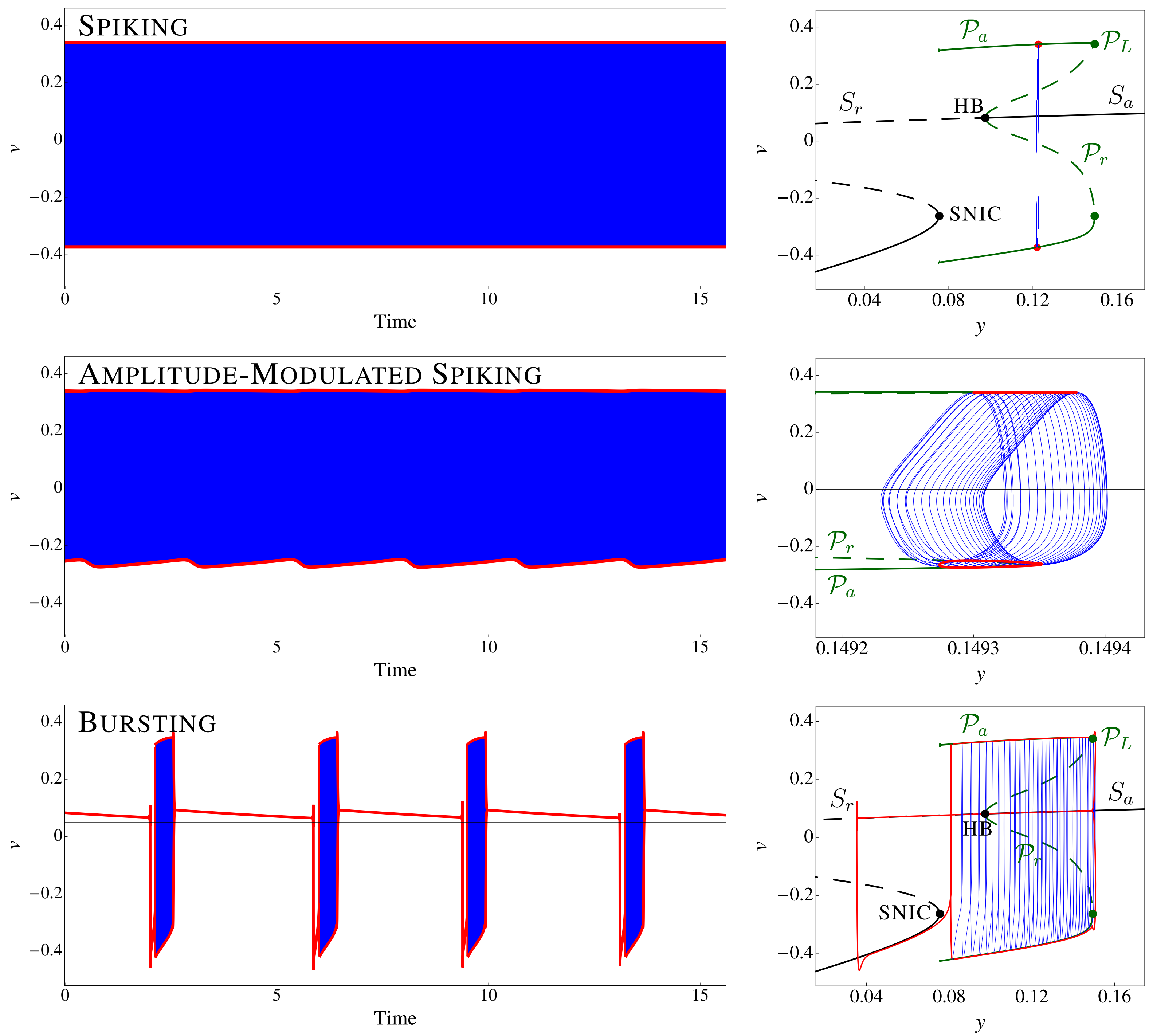
$\omega = \mathcal{O}(\sqrt{\varepsilon})$

Wang, Vo & Kaper (2016) arXiv:1607.02205

Why do we care? Reason 2

# Torus Canards & The Spiking/Bursting Transition

Attractors of Morris-Lecar with (slow) feedback current control, $y$



# Why do we care? Reason 3

## Small-Amplitude, Large-Amplitude & Mixed-Mode Oscillations
## The Curves of Maximal Canards Organize the Parameter Plane

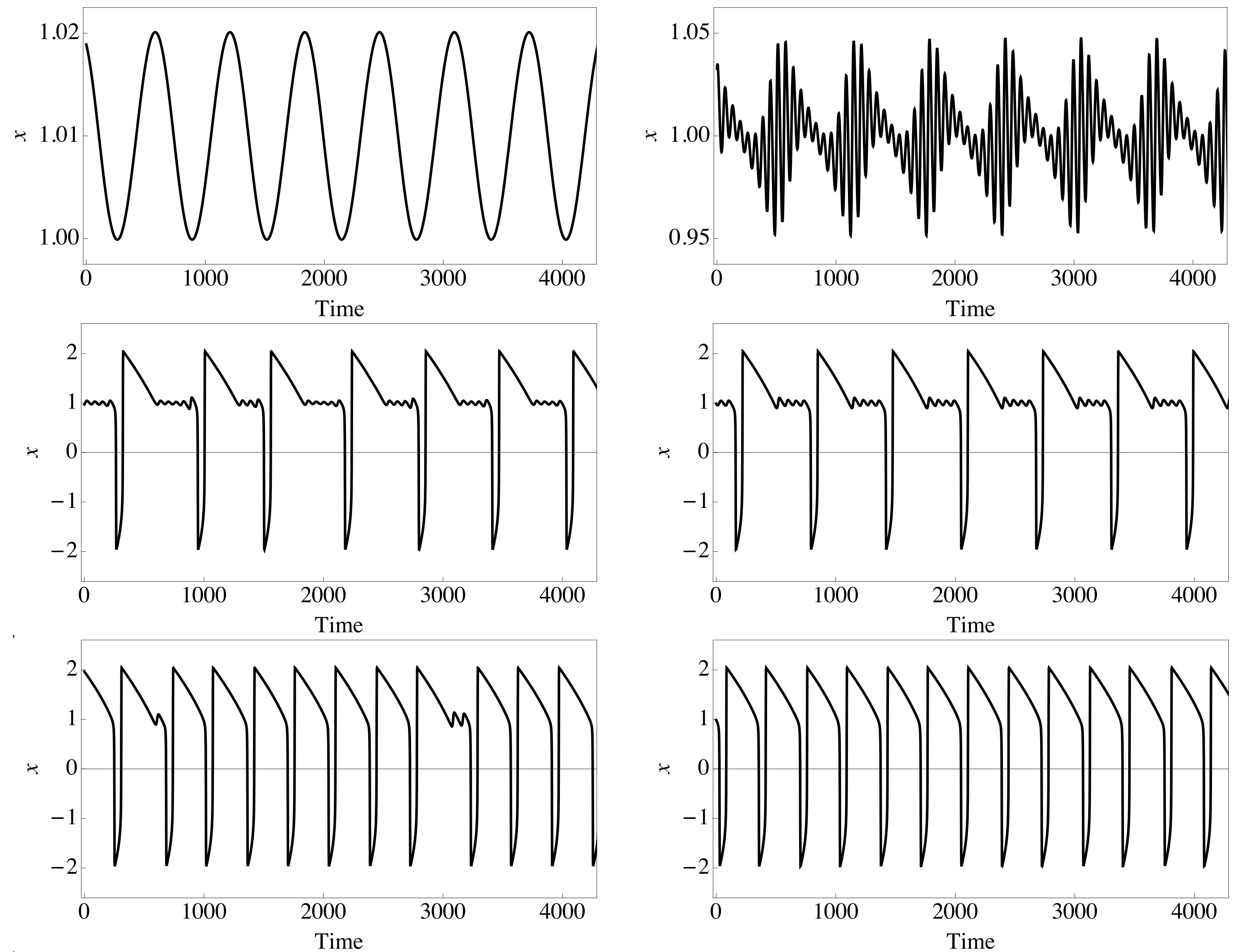
Forced van der Pol

$$\dot{x} = y - \left(\frac{x^3}{3} - x\right)$$

$$\dot{y} = \varepsilon\,(-x + a + b\cos\theta)$$

$$\dot{\theta} = \omega$$

$a \approx 1 \qquad b$ small

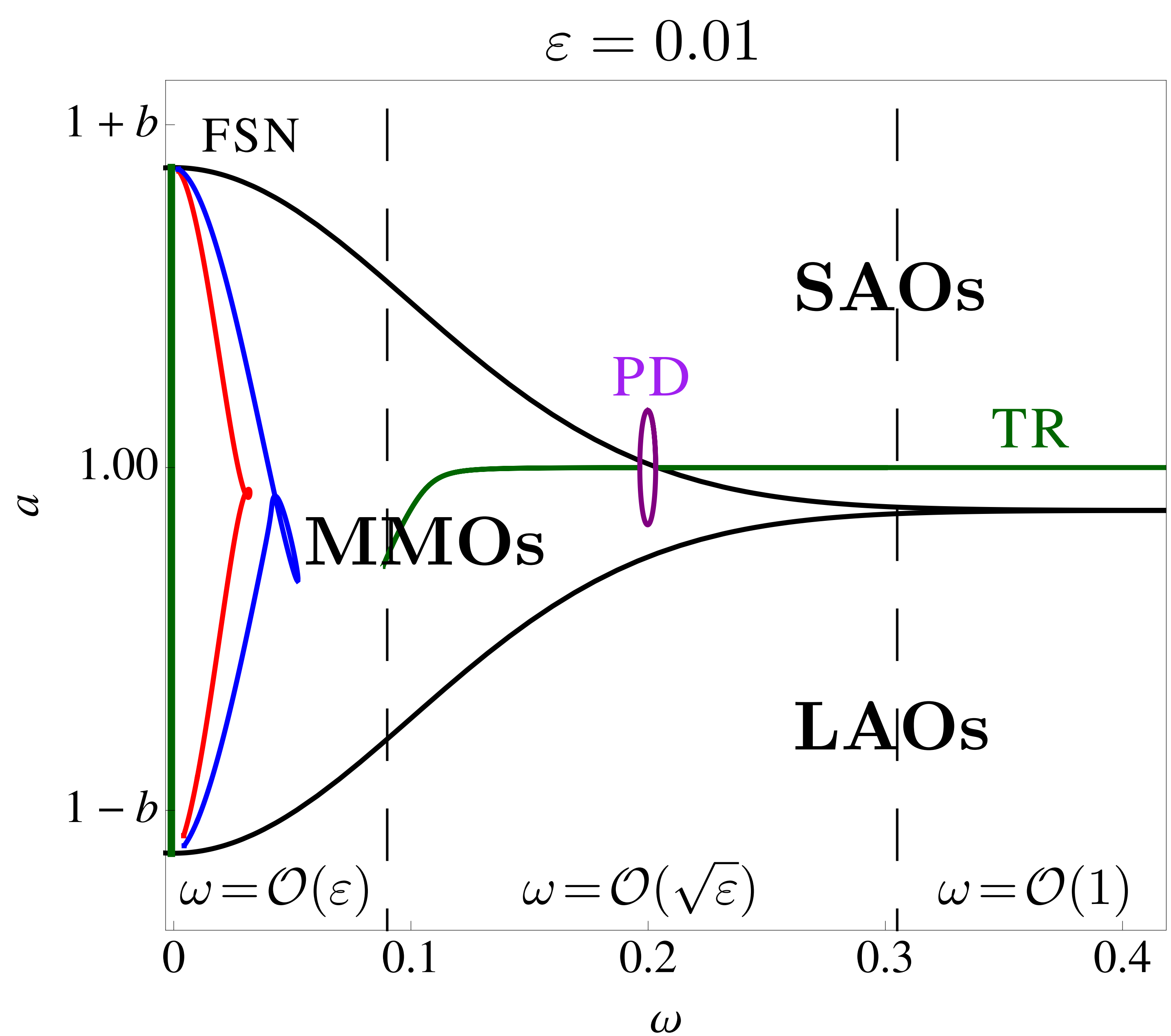


## Small-Amplitude, Large-Amplitude & Mixed-Mode Oscillations
## The Curves of Maximal Canards Organize the Parameter Plane

Forced van der Pol

$$\dot{x} = y - \left( \frac{x^3}{3} - x \right)$$
$$\dot{y} = \varepsilon \left( -x + a + b \cos \theta \right)$$
$$\dot{\theta} = \omega$$

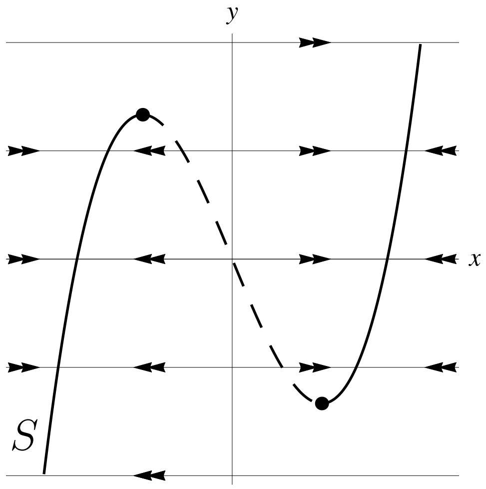
$a \approx 1 \qquad b$ small



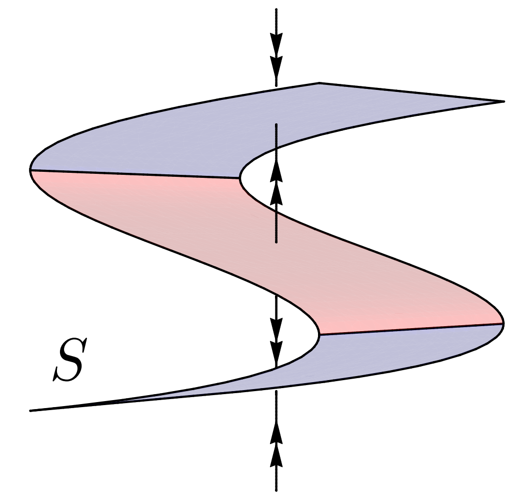

$\omega = \mathcal{O}(\varepsilon)$ ... 1-Fast/2-Slow $\qquad \omega = \mathcal{O}(\sqrt{\varepsilon})$ ... 3-Timescales $\qquad \omega = \mathcal{O}(1)$ ... 2-Fast/1-Slow

Why do we care? Reason 4

# The World of Canards




1D + Fold




2D + Fold

1-FAST/1-SLOW

**Canard Cycles**

**Singular Hopf**
**FSN II**

1-FAST/2-SLOW

**Folded Singularity Canards**

**Averaging**

**FSN I & 3-Timescales**

**Averaging**

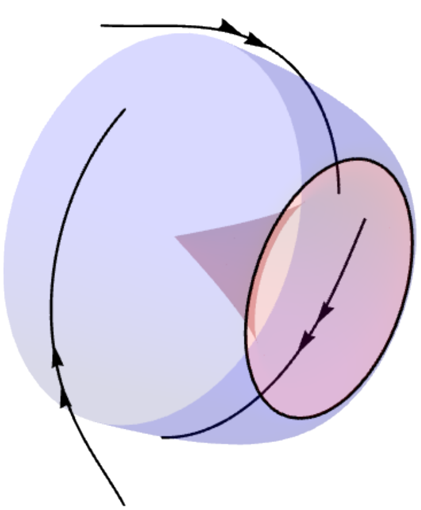
2-FAST/1-SLOW

**Degenerate Torus Canards**

**Singular Torus Bifn**

2-FAST/2-SLOW

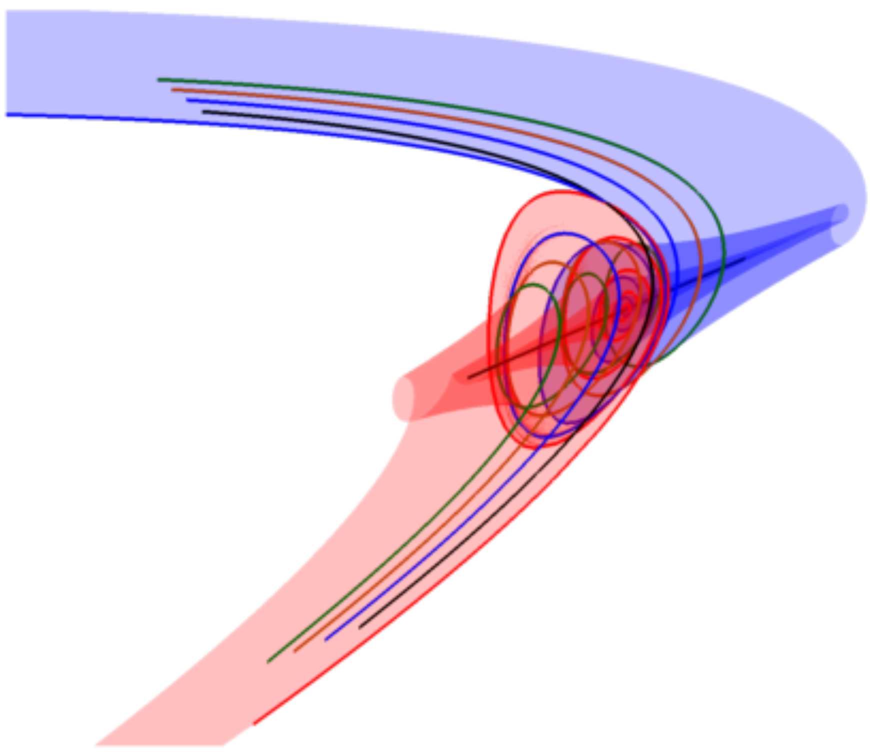
**Generic Torus Canards**

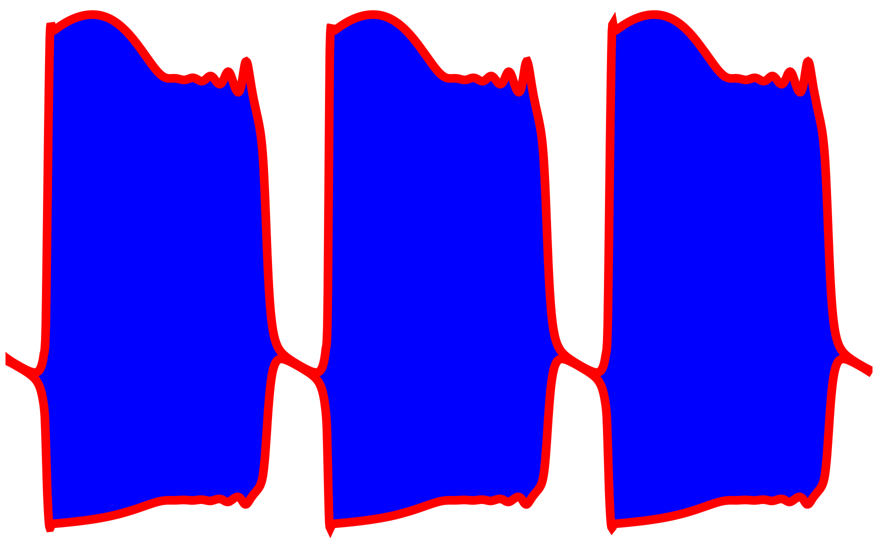


2D + SNPO

# Generic Torus Canards

- **Theoretical Contribution**
  - Averaging method for folded manifolds of limit cycles
  - Topological classification of torus canards
- **Numerical Contribution**
  - Twisted invariant manifolds of limit cycles
- **New Phenomena**
  - Genericity: should occur in models and experiments
  - Neural: amplitude-modulated bursting
  - Dynamic: torus canard-induced mixed-mode oscillations





Vo (2016) arXiv:1606.02366

# Conclusion

- **Forced van der Pol equation**
  - Existence of FSN I canards in the low-frequency regime ($\omega = \mathcal{O}(\varepsilon)$)
  - Existence of torus canards in the intermediate-frequency regime ($\omega = \mathcal{O}(\sqrt{\varepsilon})$) and high-frequency ($\omega = \mathcal{O}(1)$) regime
  - Direct connection of primary strong canards to maximal torus canards
  - Primary and secondary canard curves organize parameter plane for SAO, LAO, and MMO
- **Generalized to a class of forced slow-fast systems, including fFHN**
- **Central role in neuroscience and electrical engineering: torus canards must arise between spiking (attracting limit cycles) and bursting (of many types)**
- **Generic torus canards (2 or more slow variables) and the new phenomenon of amplitude-modulated bursting**

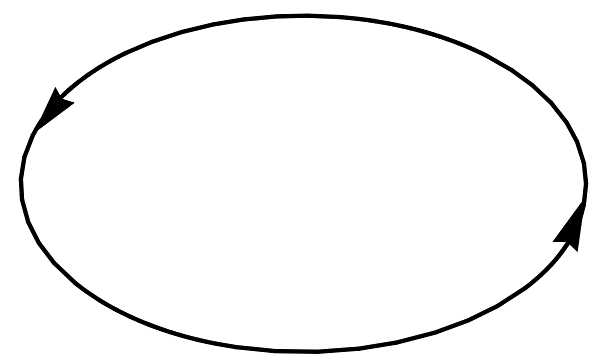
# Torus Bifurcation

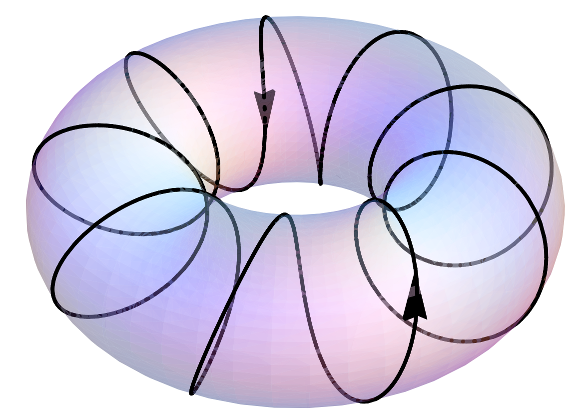
Forced van der Pol

$$\dot{x} = y - \left(\frac{x^3}{3} - x\right)$$

$$\dot{y} = \varepsilon\,(-x + a + b\cos\theta)$$

$$\dot{\theta} = \omega$$



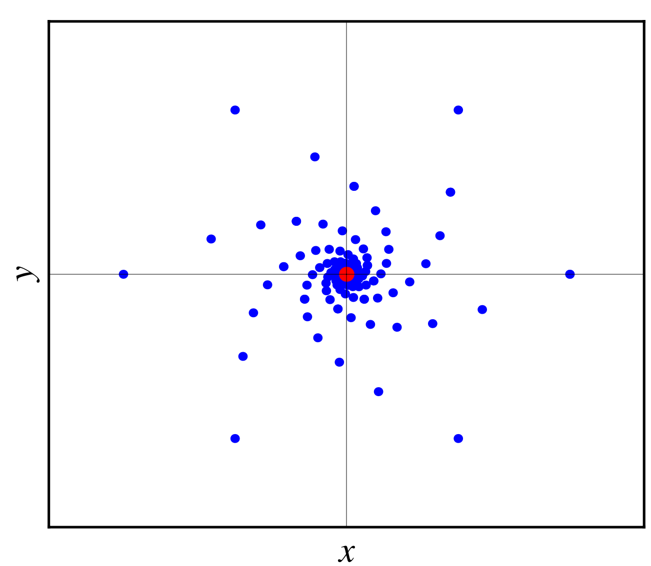


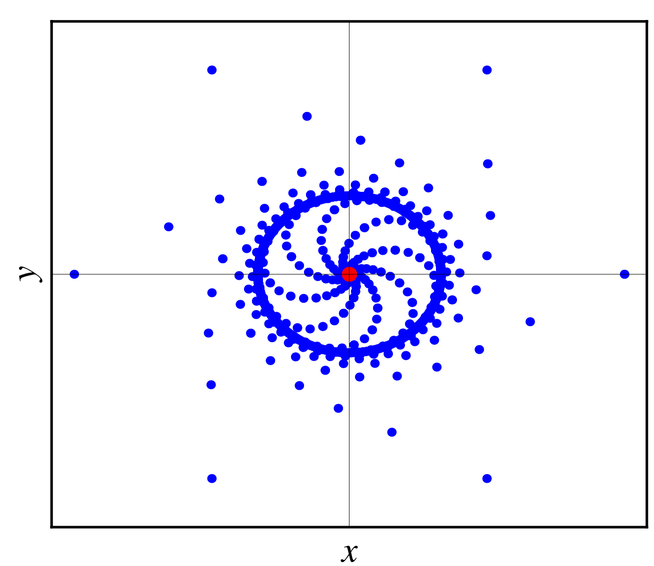
Rescaled Averaged

$$\dot{u} = v - u^2 - \frac{1}{3}\sqrt{\varepsilon}\,u^3$$

$$\dot{v} = -u + \alpha$$

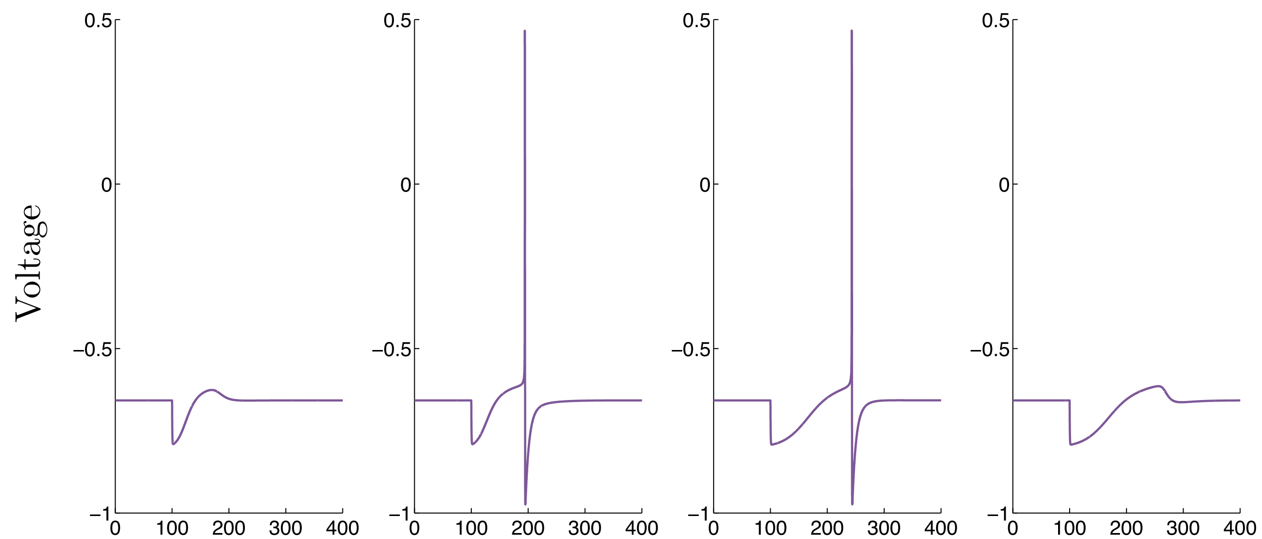
Stroboscopic Map

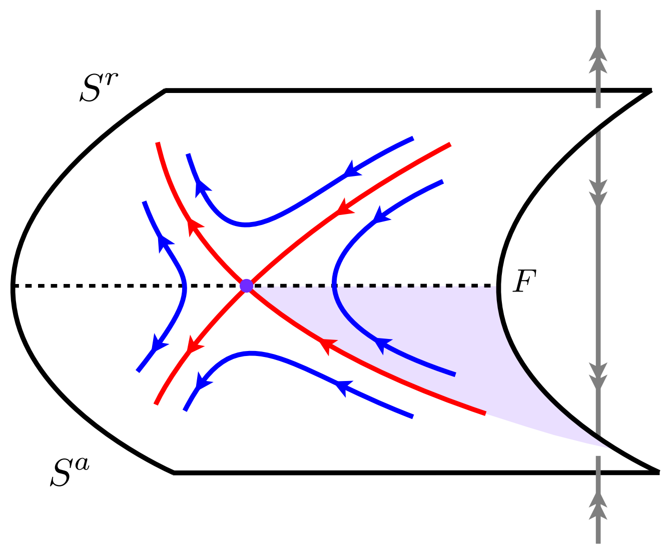


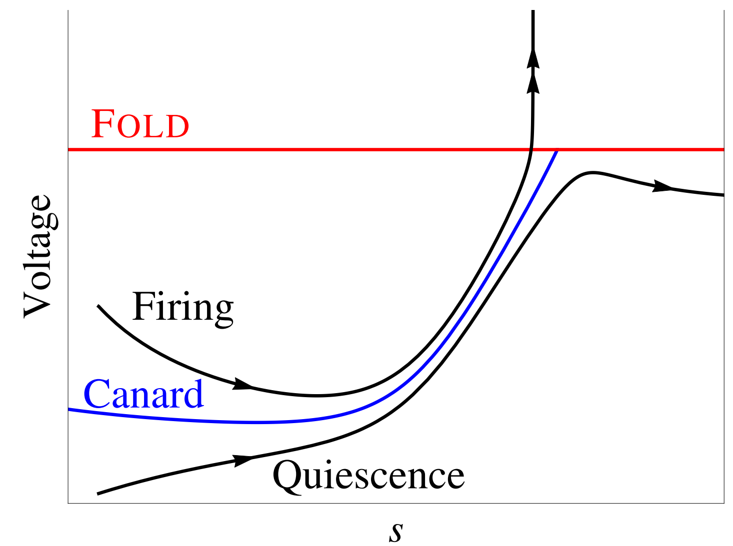



# Folded Saddle Canards

$$\dot{x} = f(x, y_1, y_2) \qquad x \;\ldots\; \textsc{Fast}$$
$$\dot{y}_1 = \varepsilon\, g_1(x, y_1, y_2) \qquad y_1 \;\ldots\; \textsc{Slow}$$
$$\dot{y}_2 = \varepsilon\, g_2(x, y_1, y_2) \qquad y_2 \;\ldots\; \textsc{Slow}$$




Mitry, McCarthy, Kopell, & Wechselberger. (2013) J. Math. Neuro., **3**:12