Almost affine copies of the Julia sets in the Mandelbrot set

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Quadratic Maps

- A quadratic map: $P = P_c : \mathbb{C} \to \mathbb{C}$, $P_c(z) = z^2 + c$ $(c \in \mathbb{C})$
- We consider its iteration:

$$\mathbb{C} \xrightarrow{P} \mathbb{C} \xrightarrow{P} \mathbb{C} \xrightarrow{P} \mathbb{C} \xrightarrow{P} \mathbb{C} \xrightarrow{P} \cdots \text{ “dynamics”}$$

$$z_0 \overset{P}{\longmapsto} z_1 \overset{P}{\longmapsto} z_2 \overset{P}{\longmapsto} z_3 \overset{P}{\longmapsto} z_4 \overset{P}{\longmapsto} \cdots \text{ “orbit”}$$

- Set $P^n := P \circ \cdots \circ P$ (n-times). Then $z_n = P^n(z_0)$.

- **Question**

  For small $\epsilon \neq 0$, how different the orbits $\{P^n(z_0)\}_{n \geq 0}$ and $\{P^n(z_0 + \epsilon)\}_{n \geq 0}$ are??

- The orbit $\{P^n(z_0)\}_{n \geq 0}$ can be “stable” or “chaotic”.

Quadratic Julia sets

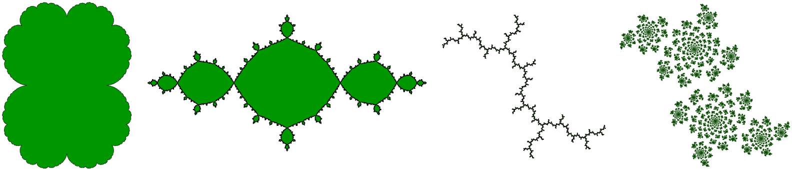
- $P_c(z) := z^2 + c$ ($c \in \mathbb{C}$), $P_c^n := P_c \circ \cdots \circ P_c$ (n times)
- A common property:

  $|z_0| > 2 + |c| \implies |P_c^n(z_0)| \to \infty$ ($n \to \infty$).
- We define the *filled Julia set* $K(P_c)$ of P_c by:

Definition

- $z_0 \in K(P_c) :\iff \{P_c^n(z_0)\}_n$ is bounded.
- $z_0 \notin K(P_c) \iff |P_c^n(z_0)| \to \infty$
- $J(P_c) := \partial K(P_c)$: the *Julia set* of P_c.

The Dichotomy



The orbit of 0 (the critical point) determines connectedness of $K(P_c)$ and $J(P_c)$.

Theorem (Julia/Fatou)

(1) $K(P_c)$ *is connected* $\iff 0 \in K(P_c)$

(2) $K(P_c)$ *is a Cantor set* $\iff 0 \notin K(P_c)$

When (2) we have $K(P_c) = J(P_c)$.

The Mandelbrot set

Definition

$c \in \mathbb{M}$: the *Mandelbrot set*

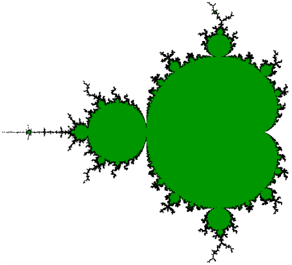
$:\iff \quad c \in \mathbb{C}$ such that $K(P_c)$ is connected.

$\iff \quad c \in \mathbb{C}$ such that $0 \in K(P_c)$.

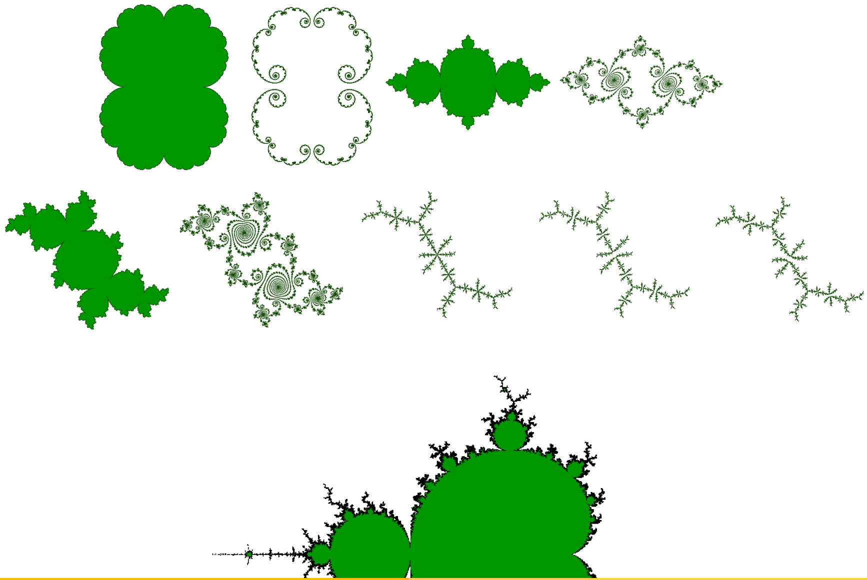
$:\iff \quad \{P_c^n(0)\}_n = \left\{0,\, c,\, c+c^2,\, (c^2+c)^2+c,\, \cdots\right\}$ is bounded.

Our result is:

$\partial\mathbb{M}$ contains super-fine copies of the Cantor-type Julia set $J(P_c)$ for various c.



The Cantor Julia sets in $\mathbb{M}$ / Movies



Quasiconformal mappings

- $h : \mathbb{C} \to \mathbb{C}$, a ori. pres. diffeo.
- Let $A = h_z(z)$ and $B = h_{\overline{z}}(z)$, then

$$h(z + \Delta z) = h(z) + A\Delta z + B\overline{\Delta z} + o(\Delta z) \quad (\Delta z \to 0)$$

- h is *conformal* at $z \iff B \equiv 0$.
- h is ori. pres. $\iff |B/A| < 1$.
- Set $\mu_h(z) := h_{\overline{z}}(z)/h_z(z) = B/A$. The quantity

$$\frac{|A| + |B|}{|A| - |B|} = \frac{1 + |\mu_h|}{1 - |\mu_h|}$$

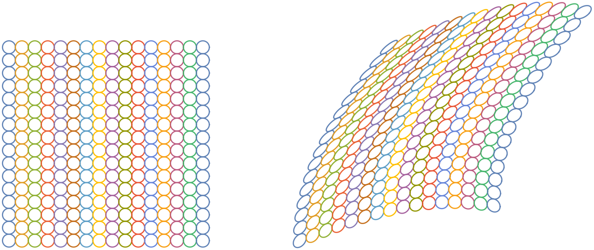
measures the distortion of h.

Quasiconformal mappings (2)

Definition

An ori. pres. homeo. $h : \mathbb{C} \to \mathbb{C}$ is K*-quasiconformal* if

1. h is "ACL" ($\Longrightarrow$ totally differentiable a.e.)
2. $\dfrac{1 + |\mu_h|}{1 - |\mu_h|} \leq K$ a.e.



Y appears quasiconformally in X

Definition

X, Y : compact subsets of $\mathbb{C}$, $K \geq 1$.

X contains a K-quasiconformal copy of Y

$:\iff$ $^{\exists}$ an embedding $\phi : Y \to X$ s.t.

1. $\phi(\partial Y) \subset \partial X$
2. ϕ extends to a K-qc $\tilde{\phi} : \mathbb{C} \to \mathbb{C}$.

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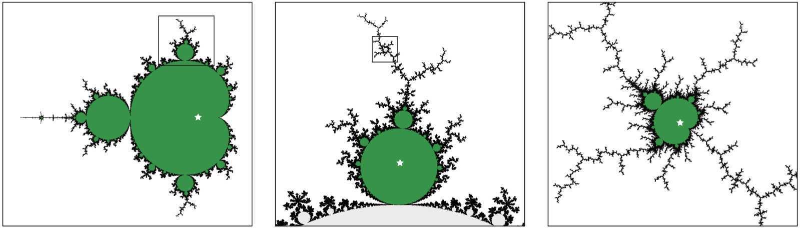
X appears quasiconformally in Y: Example

Example: $\mathbb{M}$ contains a proper K-quasiconformal copy of $\mathbb{M}$ itself for any $K > 1$ (Douady-Hubbard-Lyubich).



The main results

Theorem (K-Kisaka)

For any choices of $\begin{cases} c_0 \in \partial\mathbb{M} \\ \epsilon > 0 \\ \kappa > 0 \end{cases}$ *, there exists a*

$$\sigma \in (\mathbb{C} - \mathbb{M}) \cap \mathbb{D}(c_0, \epsilon)$$

such that $\mathbb{M}$ *contains a* $(1+\kappa)$*-quasiconformal copy of the Cantor Julia set* $J(P_\sigma)$*.*

Remark: Such a copy is ubiquitous in $\partial\mathbb{M}$.

Remark2: This extends a result by Douady, Buff, Devaney, and Sentenac for $c_0 = 1/4 \in \partial\mathbb{M}$, without estimate of the distortion [2000].

The main results (2)

Theorem (K)

For the same parameter σ *as above, we can find an embedding* $\phi : J(P_\sigma) \to \partial\mathbb{M}$ *such that*

- *the distortion (the "*$K > 1$*") of* ϕ *is arbitrarily close to one.*
- *the inverse* ϕ^{-1} *is arbitrarily close to an affine map.*

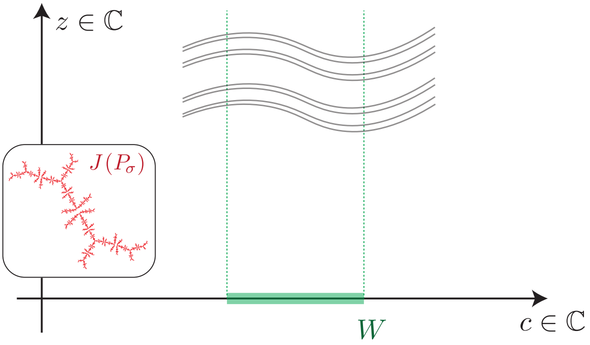
That is, $^\forall \epsilon > 0,$ $^\forall \kappa > 0,$ $^\exists$ $(1+\kappa)$*-qc embedding* $\phi : J(P_\sigma) \to \partial\mathbb{M}$ *and* $^\exists$ *cpx. affine map* $A(z) = \alpha z + \beta$ *s.t.*

$$\max_{z \in \phi(z)} |\phi^{-1}(z) - (\alpha z + \beta)| < \epsilon.$$

Idea of the Proof

$^{\exists}\ \phi : J(P_\sigma) \times W \to \mathbb{C}$ s.t. by setting $\phi(z, c) = \phi_c(z)$, we have:

- ϕ_c is a $(1 + \kappa)$-qc embedding of $J(P_\sigma)$ into $J(P_c)$.
- $^{\exists}\ \ell$ s.t. $^{\forall} c \in W$, $P_c^\ell(J_c^*) = J_c^*$ where $J_c^* := \phi_c(J(P_\sigma))$.
- $^{\forall}\ z \in J(P_\sigma)$, $c \mapsto \phi_c(z)$ is a holomorphic function on W.



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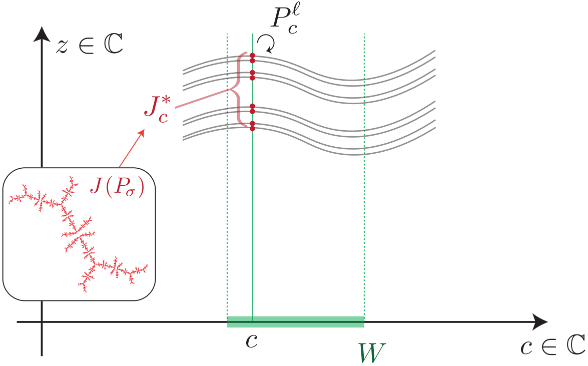

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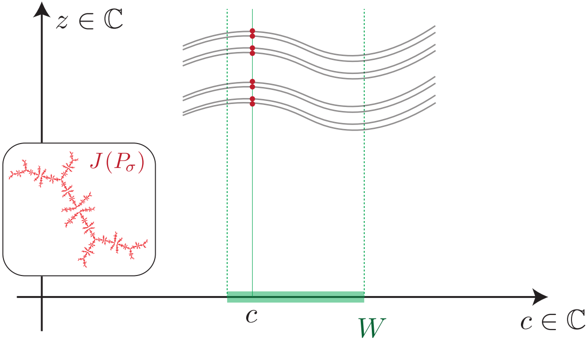

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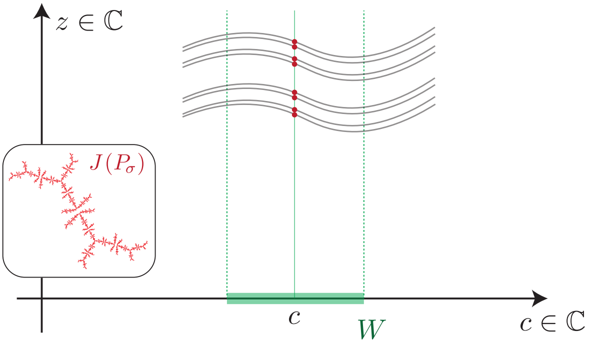

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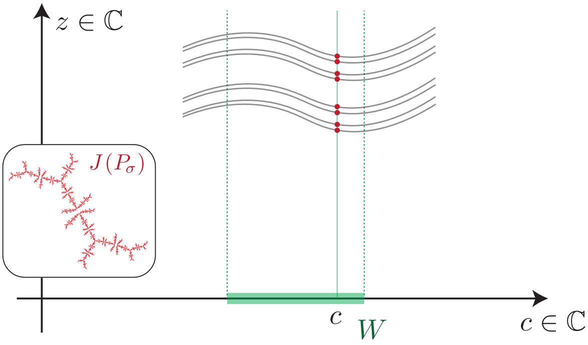

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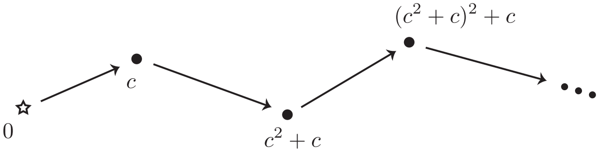

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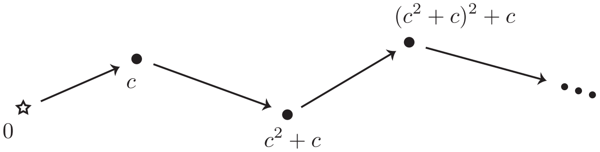

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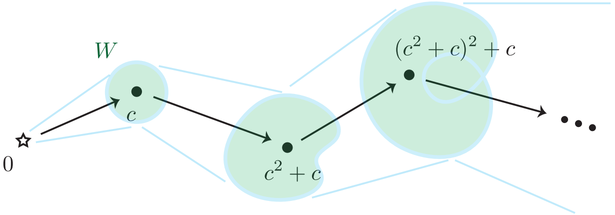

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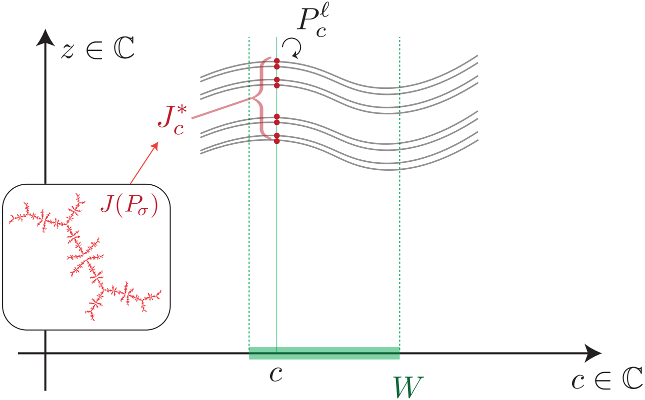

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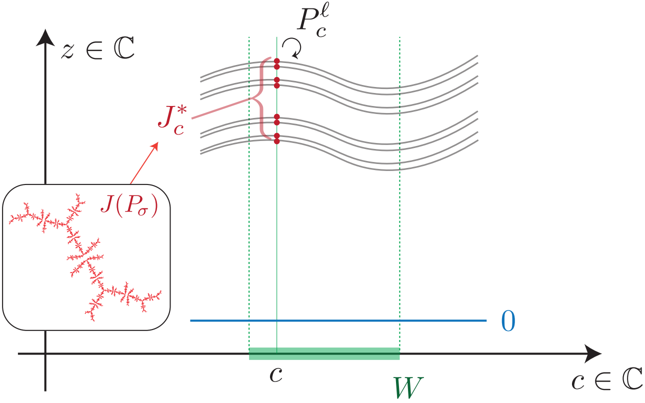

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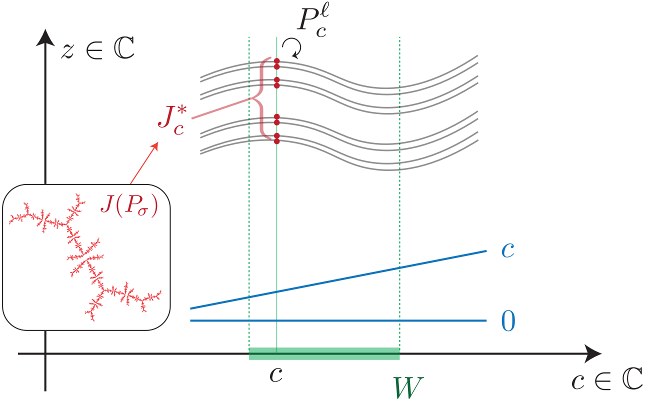

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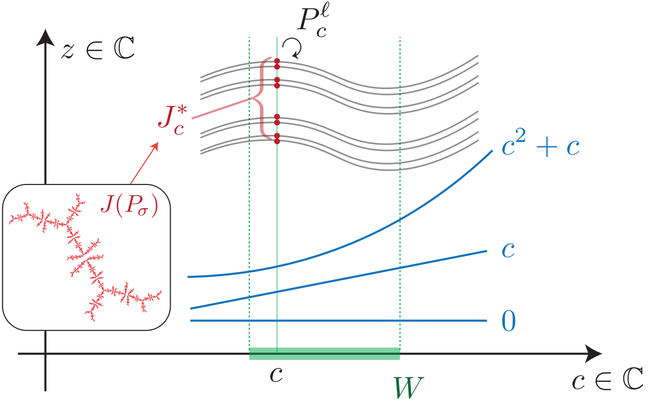

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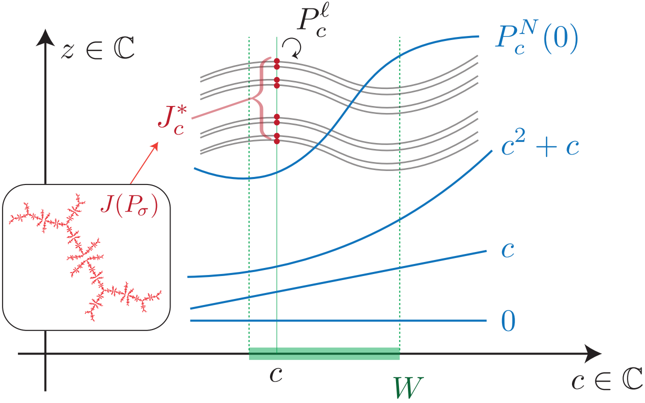

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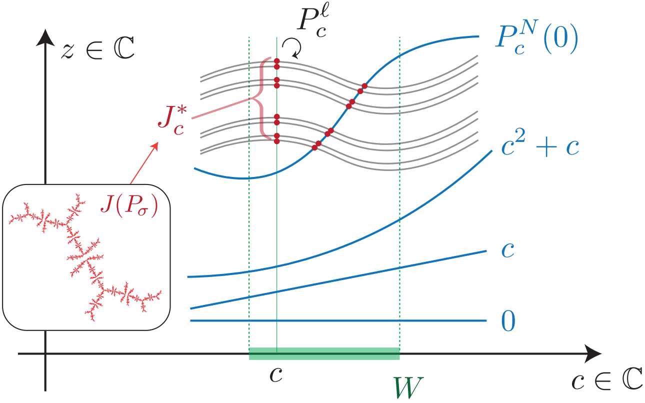

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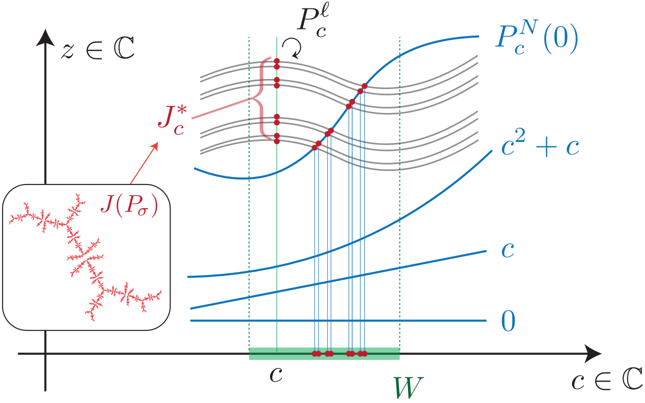

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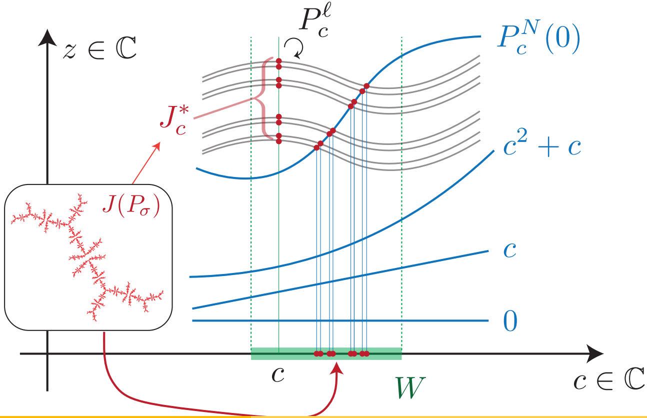

–Thank you! –

References

T. Kawahira and M. Kisaka. Julia sets appear quasiconformally in the Mandelbrot set, *Preprint*, arXiv:1804.00176.

Movies available at my web page.

http://www.math.titech.ac.jp/~kawahira