Area preserving surface diffeomorphisms with polynomial decay of correlations are ubiquitous

Farruh Shahidi
Joint work with Ya. Pesin and S.Senti

The Pennsylvania State University

June 25-29, 2018
Boston-Keio Workshop

Plan

- 1 Smooth realization problem
- 2 Preliminaries and the main result
- 3 The main result on the 2-torus
- 4 Proof scheme
- 5 Extension to any surface

smooth realization problem

A classical problem in smooth dynamics known as the *smooth realization problem* asks whether there is a diffeomorphism f of a compact smooth manifold M which has a prescribed collection of ergodic properties with respect to a *natural* invariant measure μ such as the Riemannian volume.

A more interesting but substantially more difficult version of the smooth realization problem is to construct a volume preserving diffeomorphism f with prescribed ergodic properties on any given smooth manifold M.

Some results in smooth realization problem

- Anosov and Katok (1970) constructed an example of a volume preserving ergodic C^∞ map.
- Katok (1979) gave an example of area preserving C^∞ diffeomorphism with non-zero Lyapunov exponents on any surface which is Bernoulli.
- Brin, Feldman, and Katok (1981) have later extended this result by constructing a volume preserving C^∞ diffeomorphism, which is Bernoulli, on any Riemannian manifold of dimension ≥ 5. In their example the map has all but one non-zero Lyapunov exponents.
- Finally, Dolgopyat and Pesin (2002) provided an example of a volume preserving C^∞ Bernoulli diffeomorphism with non-zero Lyapunov exponents on any Riemannian manifold of dimension ≥ 2.

Question: Does a compact manifold admit a volume preserving diffeomorphism with non-zero Lyapunov exponents that enjoys other important statistical properties such as exponential or polynomial decay of correlations, the Central Limit Theorem, and the Large Deviations property?
In polynomial case, the answer is positive on every connected compact surface.

Preliminaries

Let X be a measurable space and $T : X \to X$ a measurable invertible transformation preserving a measure μ. We recall some definitions.

Decay of correlations. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two classes of observables on X. For $h_1 \in \mathcal{H}_1$ and $h_2 \in \mathcal{H}_2$ define the correlation function

$$\mathrm{Cor}_n(h_1, h_2) := \int h_1(T^n(x))h_2(x)\, d\mu - \int h_1(x)\, d\mu \int h_2(x)\, d\mu.$$

We say that T has *polynomial decay of correlations* with respect to classes $\mathcal{H}_1$ and $\mathcal{H}_2$ if there exist $\gamma' > 0,\ C' > 0$ such that for any $h_1 \in \mathcal{H}_1$, $h_2 \in \mathcal{H}_2$, and any $n > 0$,

$$|\mathrm{Cor}_n(h_1, h_2)| \leq C' n^{-\gamma'}$$

Moreover, if

$$|\mathrm{Cor}_n(h_1, h_2)| \geq C n^{-\gamma},$$

for $\gamma,\ C > 0$ then the map T admits a *polynomial lower bound for correlations.*

Preliminaries

The Central Limits Theorem. We say that T satisfies the *Central Limit Theorem (CLT)* for a class $\mathcal{H}$ of observables on X if there exists $\sigma > 0$ such that for any $h \in \mathcal{H}$ with $\int h = 0$ the sum

$$\frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} h(f^i(x))$$

converges in law to a normal distribution $N(0, \sigma)$.

Preliminaries

The Large Deviation property. We say that T has the *Polynomial Large Deviation property* with respect to a class $\mathcal{H}$ of observables on X if for any $\delta > 0$, $\varepsilon > 0$, and any $h \in \mathcal{H}$ there exists $C = C(\delta, \varepsilon, h) > 0$ such that for all n

$$\mu\Big(\Big|\frac{1}{n}\sum_{i=0}^{n-1} h(f^i(x)) - \int h\Big| > \varepsilon\Big) < Cn^{-\beta},$$

where $\beta > 0$ is a constant independent of δ, ε, and h.

Main result

Let M be a smooth compact connected surface and m the area. Without loss of generality we assume that $m(M) = 1$. Let $\mathcal{H}_1 := L^\infty(M, m)$ and let $\mathcal{H}_2 := C^\alpha(M)$ be the class of all Hölder continuous functions on M.
Consider a nested sequence of subsets $\{M_j\}$ that exhaust M that is $M_1 \subset M_2 \subset \ldots \subset M$ and $\bigcup_{j\geq 1} M_j = M$. Given such a sequence, for $i = 1, 2$, let $\mathcal{G}_i = \mathcal{G}_i(\{M_j\})$ be the class of observables $h_i \subset \mathcal{H}_i$ for which there is $k = k(h_i)$ such that $\mathrm{supp}(h_i) \subset M_k$.
Given $0 < \alpha < \frac{1}{6}$ and $0 < \mu < 1$, denote by

$$\gamma = \frac{1}{2\alpha} + \frac{1-\mu}{2}, \quad \gamma' = \frac{1}{2\alpha} + \frac{1-\mu}{2^{\alpha+2}}. \tag{1}$$

Observe that $\gamma > 3$ and $\gamma' > 3$. Set $\kappa = \frac{\alpha}{1-\alpha}$.

Theorem 1

For any $0 < \alpha < \frac{1}{6}$ there is an area preserving $C^{1+\kappa}$ diffeomorphism f of M satisfying:

1. f has the Bernoulli property;
2. f has non-zero Lyapunov exponents with respect to m;
3. f has polynomial decay of correlations with respect to the classes $\mathcal{H}_1$ and $\mathcal{H}_2$ of observables and admits a polynomial lower bound for correlations with respect to some sequence of subsets $\{M_j\}$ and the corresponding classes $\mathcal{G}_1$ and $\mathcal{G}_2$ of test functions; more precisely,
   - 3.1 if $h_i \in \mathcal{H}_i$, $i = 1, 2$ and $\int h_1\, dm \int h_2\, dm \neq 0$, then
   $$|\mathrm{Cor}_n(h_1, h_2)| \leq C' n^{-(\gamma'-3)},$$
   where $C' = C'(\|h_1\|_{L^\infty}, \|h_2\|_{C^\alpha})$;
   - 3.2 if $h_i \in \mathcal{G}_i$, $i = 1, 2$ and $\frac{1}{7} < \alpha < \frac{1}{6}$, then
   $$Cn^{-(\gamma-2)} \leq \mathrm{Cor}_n(h_1, h_2),$$
   where $C = C(\|h_1\|_{L^\infty}, \|h_2\|_{C^\alpha})$;

Main result

4.f satisfies the CLT for the class of observables $h \in \mathcal{H}_2$, $\int h = 0$ with $\sigma = \sigma(h)$ given by

$$\sigma^2 = -\int h^2 dm + 2\sum_{n=0}^{\infty} \int h \cdot h \circ f^n dm,$$

where $\sigma > 0$ if and only if h is not cohomologous to zero, i.e. $h \circ f \neq g \circ f - g$ for any g;
5. f has the Polynomial Large Deviation property with respect to the class $\mathcal{H}_2$ of observables with the constant C of the form $C = C(\|h\|_{C^\alpha})\varepsilon^{-2(\gamma'-3-\delta)}$ and $\beta = \gamma' - 3 - \delta$ for some sufficiently small $\delta > 0$. In addition, for an open and dense subset of observables in $\mathcal{H}_2$ and sufficiently small $\varepsilon > 0$

$$\frac{1}{n^{\gamma'-3+\delta}} < m(|\frac{1}{n}\sum_{i=0}^{n-1} h(f^i(x)) - \int h| > \varepsilon)$$

for infinitely many $n's$.

An example on 2− torus

Consider the automorphism of the two-dimensional torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ given by the matrix $A := \left(\begin{smallmatrix} 5 & 8 \\ 8 & 13 \end{smallmatrix}\right)$. It has four fixed points $x_1 = (0,0)$, $x_2 = (\frac{1}{2}, 0)$, $x_3 = (0, \frac{1}{2})$, and $x_4 = (\frac{1}{2}, \frac{1}{2})$. For $i = 1, 2, 3, 4$ consider the disk $D_r^i = \{(s_1, s_2) : {s_1}^2 + {s_2}^2 \le r^2\}$ of radius r centered at x_i and set $D_r = \bigcup_{i=1}^4 D_r^i$. Here (s_1, s_2) is the coordinate system obtained from the eigendirections of A and originated at x_i. Let $\lambda > 1$ be the largest eigenvalue of A. There are $r_1 > r_0$ such that

$$D_{r_0}^i \subset \mathrm{int} A(D_{r_1}^i) \cap \mathrm{int} A^{-1}(D_{r_1}^i) \tag{2}$$

and the disks $D_{r_1}^i$ are pairwise disjoint. Fix i and consider the system of differential equations in $D_{r_1}^i$

$$\frac{ds_1}{dt} = s_1 \log \lambda, \quad \frac{ds_2}{dt} = -s_2 \log \lambda. \tag{3}$$

Observe that $A|_{D_{r_0}^i}$ is the time-1 map of the local flow generated by this system.

slow-down

We choose a number $0 < \alpha < 1$ and a function $\psi : [0,1] \mapsto [0,1]$ satisfying:

(K1) ψ is of class C^∞ everywhere but at the origin;

(K2) $\psi(u) = 1$ for $u \geq r_0$ and some $0 < r_0 < 1$;

(K3) $\psi'(u) > 0$ for $0 < u < r_0$;

(K4) $\psi(u) = (u/r_0)^\alpha$ for $0 \leq u \leq \frac{r_0}{2}$.

Using the function ψ, we slow down trajectories of the flow by perturbing the system (3) in $D^i_{r_0}$ as follows

$$\begin{aligned} \frac{ds_1}{dt} &= \quad s_1\psi({s_1}^2 + {s_2}^2)\log\lambda \\ \frac{ds_2}{dt} &= - s_2\psi({s_1}^2 + {s_2}^2)\log\lambda. \end{aligned} \tag{4}$$

slow-down

This system of differential equations generates a local flow. Denote by g^i the time-1 map of this flow. The choices of ψ, r_0 and r_1 (see (2)) guarantee that the domain of g^i contains $D^i_{r_0}$. Furthermore, g^i is of class C^∞ in $D^i_{r_0} \setminus \{x_i\}$ and it coincides with A in some neighborhood of the boundary $\partial D^i_{r_0}$. Therefore, the map

$$G(x) = \begin{cases} A(x) & \text{if } x \in \mathbb{T}^2 \setminus D_{r_0}, \\ g^i(x) & \text{if } x \in D^i_{r_0} \end{cases} \tag{5}$$

defines a homeomorphism of the torus $\mathbb{T}^2$, which is a C^∞ diffeomorphism everywhere except at the fixed points x_i.

slow-down

Since $0 < \alpha < 1$, we have that

$$\int_0^1 \frac{du}{\psi(u)} < \infty.$$

This implies that the map G preserves the probability measure $d\nu = \kappa_0^{-1} \kappa \, dm$ where m is the area and the density κ is a positive C^∞ function that is infinite at x_i and is defined by

$$\kappa(s_1, s_2) := \begin{cases} (\psi({s_1}^2 + {s_2}^2))^{-1} & \text{if } (s_1, s_2) \in D^i_{r_0}, \\ 1 & \text{in } \mathbb{T}^2 \setminus D_{r_0} \end{cases}$$

slow-down

We further perturb the map G by a coordinate change ϕ in $\mathbb{T}^2$ to obtain an area-preserving map. To achieve this, define a map ϕ in $D^i_{r_0}$ by the formula

$$\phi(s_1, s_2) := \frac{1}{\sqrt{\kappa_0({s_1}^2 + {s_2}^2)}} \left(\int_0^{{s_1}^2 + {s_2}^2} \frac{du}{\psi(u)} \right)^{1/2} (s_1, s_2) \qquad (6)$$

and set $\phi = \mathrm{Id}$ in $\mathbb{T}^2 \setminus D_{r_0}$. Clearly, ϕ is a homeomorphism and is a C^∞ diffeomorphism outside the points x_1, x_2, x_3, x_4. One can show that ϕ transfers the measure ν into the area and that the map $f_{\mathbb{T}^2} = \phi \circ G \circ \phi^{-1}$ is a homeomorphism and is a C^∞ diffeomorphism outside the points x_1, x_2, x_3, x_4. It is called a *slow down map*.

Properties of the slow-down map

The following proposition describes some basic properties of this map.

Proposition

The map $f_{\mathbb{T}^2}$ has the following properties:

1. *It is topologically conjugated to A via a homeomorphism H.*
2. *It is ergodic with respect to the area m.*
3. *$f_{\mathbb{T}^2}$ is of class of smoothness $C^{2+\kappa}$ where $\kappa = \frac{\alpha}{1-\alpha}$.*

Theorem 2

The map $f_{\mathbb{T}^2}$ has the properties stated in Theorem 1.

Proof steps

The proof consists of the following steps.

- Represent the map $f_{\mathbb{T}^2}$ as Young's diffeomorphism and tower.
- Estimate the tail of the return time function.
- Use general result by Gouëzel to obtain estimates for correlation function and the CLT.
- Use Melbourne and Nicol's results to obtain polynomial large deviation estimate.

Tower representation for a slow-down map

Let $\Lambda = \cup_i \Lambda_i$ and $F : \Lambda \to \Lambda$ is given by $F(x) = f^{\tau_i}(x)$ where $x \in \Lambda_i$ and τ_i is the return time of Λ_i to Λ. We define the *Young tower* with the base Λ by setting:
$\hat{Y} = \{(x, k) \in \Lambda \times \mathbb{N} : 0 \leq k < \tau(x)\}$ and the tower map $\hat{f} : \hat{Y} \to \hat{Y}$ defined by

$$\hat{f}(x, k) = \begin{cases} (x, k+1) & \textit{if } \ k < \tau(x) - 1 \\ (Fx, 0) & \textit{if } \ k = \tau(x) - 1. \end{cases}$$

Define the $\hat{f}-$ probability invariant measure $\hat{m} = m \times \textit{counting}/(\int_\Lambda \tau)$.

Gouëzel's Theorem

Assume $(\hat{Y}, \hat{m}, \hat{f})$ is the Young tower with $gcd\{\tau_i\} = 1$ and $\hat{m}(\tau > n) = \mathcal{O}(\frac{1}{n^\beta}), \beta > 1$. Assume in addition that for some $C > 0$ and $0 < \rho < 1$.

$$\left|\frac{JacF(x)}{JacF(y)} - 1\right| < C\rho^{s(Fx,Fy)}$$

Then for $\hat{h_1} \in L^\infty(\hat{Y}),\ \hat{h_2} \in Lip(\hat{Y})$, supported inside $k-$th level of the tower for some k, one has

$$Cor(\hat{h_1} \circ \hat{f}^n, \hat{h}_2) = \sum_{n>N}^{\infty} m(\{x | \tau(x) > N\}) \int_{\hat{Y}} \hat{h_1}\, d\hat{m} \int_{\hat{Y}} \hat{h_2}\, d\hat{m} + \mathcal{O}(R_\beta(n)) \tag{7}$$

where $R_\beta(n) = \frac{1}{n^\beta}$ if $\beta > 2$; $R_\beta(n) = \frac{log\ n}{n^2}$ if $\beta = 2$ and $R_\beta(n) = \frac{1}{n^{2\beta-2}}$ if $1 < \beta < 2$.

Tail estimate

$$C\frac{1}{n^{\gamma-1}} \leq m(\{x : \tau(x) > n\}) \leq C'\frac{1}{n^{\gamma'-1}}$$

Extension to any surface

Proposition (Katok, 1979)

There exists a map $\zeta\colon \mathbb{T}^2 \to S^2$ *satisfying:*

1. ζ *is a double branched covering, is one-to-one on each branch, and* C^∞ *everywhere except at the points* x_i, $i = 1, 2, 3, 4$ *where it branches;*
2. $\zeta \circ I = \zeta$ *where* $I\colon \mathbb{T}^2 \to \mathbb{T}^2$ *is the involution map given by* $I(t_1, t_2) = (1 - t_1, 1 - t_2)$;
3. ζ *preserves area, i.e.,* $\zeta_* m = m_{S^2}$ *where* m_{S^2} *is the area in* S^2;
4. *there exists a local coordinate system in a neighborhood of each point* $\zeta(x_i)$, $i = 1, 2, 3, 4$ *in which*

$$\zeta(s_1, s_2) = \left(\frac{{s_1}^2 - {s_2}^2}{\sqrt{{s_1}^2 + {s_2}^2}}, \frac{2 s_1 s_2}{\sqrt{{s_1}^2 + {s_2}^2}} \right)$$

in each disk $D^i_{r_0}$.

It is easy to see that the map $f_{S^2} := \zeta \circ f_{\mathbb{T}^2} \circ \zeta^{-1}$ preserves area.

The sphere can be unfolded into the unit disk D^2 and the map f_{S^2} can be carried over to an area-preserving map f_{D^2} of the disk. To see this set $p_i = \zeta(x_i)$, $i = 1, 2, 3, 4$. In a small neighborhood of the point p_4 we define a map η by

$$\eta(\tau_1, \tau_2) = \left(\frac{\tau_1 \sqrt{1 - \tau_1^2 - \tau_2^2}}{\sqrt{\tau_1^2 + \tau_2^2}}, \frac{\tau_2 \sqrt{1 - \tau_1^2 - \tau_2^2}}{\sqrt{\tau_1^2 + \tau_2^2}} \right).$$

One can extend η to an area preserving C^∞ diffeomorphism (still denoted by η) between $S^2 \setminus \{p_4\}$ and the unit disk D^2. The map

$$f_{D^2} := \eta \circ f_{S^2} \circ \eta^{-1} \tag{8}$$

is a of D^2 that preserves area m_{D^2} and is identity on the boundary.

Proposition

The maps f_{S^2} and f_{D^2} are of class of smoothness $C^{2+2\kappa}$, where $\kappa = \frac{\alpha}{1-\alpha}$.

Proposition (Katok 1979)

Given a compact surface M, there exists a continuous map $\tau\colon D^2 \to M$ such that:

1. *the restriction $\tau|intD^2$ is a diffeomorphic embedding;*
2. $\tau(D^2) = M$*;*
3. τ *preserves area; more precisely, $\tau_* m_{D^2} = m_M$ where m_M is the area in M. Moreover, $m_M(M \setminus \tau(intD^2)) = 0$;*

Proposition

The map $f_M := \eta \circ f_{D^2} \circ \eta^{-1}$ is a $C^{1+\kappa}$ area preserving diffeomorphism of the surface.

Thank you!