Variational construction of periodic and connecting orbits in the planar Sitnikov problem

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1 The three-body problem

Consider the planar three-body problem which is governed by the following ODEs:

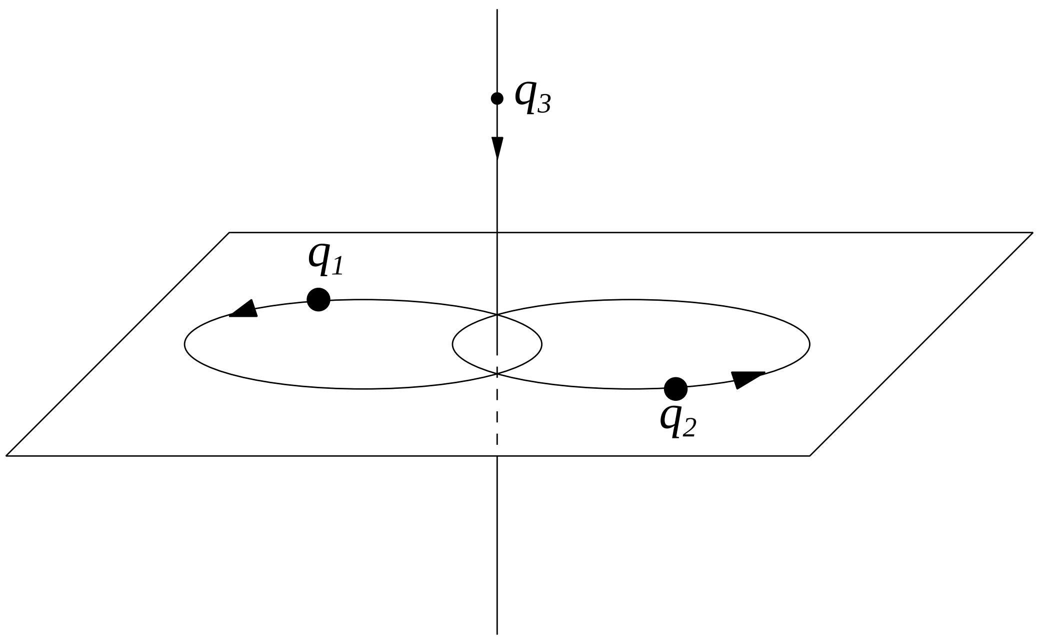
$$
\begin{aligned}
\ddot{\boldsymbol{q}}_1 &= -m_2 \frac{\boldsymbol{q}_1 - \boldsymbol{q}_2}{|\boldsymbol{q}_1 - \boldsymbol{q}_2|^3} - m_3 \frac{\boldsymbol{q}_1 - \boldsymbol{q}_3}{|\boldsymbol{q}_1 - \boldsymbol{q}_3|^3} \\
\ddot{\boldsymbol{q}}_2 &= -m_1 \frac{\boldsymbol{q}_2 - \boldsymbol{q}_1}{|\boldsymbol{q}_2 - \boldsymbol{q}_1|^3} - m_3 \frac{\boldsymbol{q}_2 - \boldsymbol{q}_3}{|\boldsymbol{q}_2 - \boldsymbol{q}_3|^3} \\
\ddot{\boldsymbol{q}}_3 &= -m_1 \frac{\boldsymbol{q}_3 - \boldsymbol{q}_1}{|\boldsymbol{q}_3 - \boldsymbol{q}_1|^3} - m_2 \frac{\boldsymbol{q}_3 - \boldsymbol{q}_2}{|\boldsymbol{q}_3 - \boldsymbol{q}_2|^3}.
\end{aligned}
\tag{1}
$$

Here $\boldsymbol{q}_k \in \mathbb{R}^3$ and $m_k \geq 0$ for $k = 1, 2, 3$ represent positions and masses.

2 Sitnikov problem

The Sitnikov problem is a special case of the three-body problem defined as following:

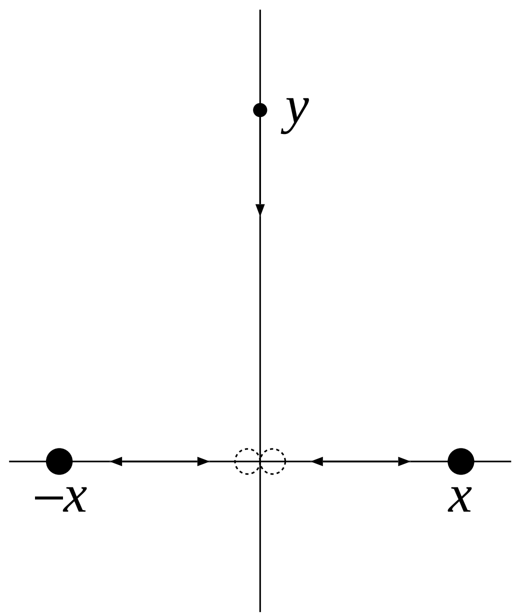
- $m_1 = m_2 = 1$ and $m_3 = 0$;
- $\boldsymbol{q}_1(t)$ and $\boldsymbol{q}_2(t)$ move on the xy-plane and satisfy the Kepler problem $\ddot{\boldsymbol{q}}_1 = -\frac{1}{4|\boldsymbol{q}_1|^3}\boldsymbol{q}_1$ $(\boldsymbol{q}_1(t) = -\boldsymbol{q}_2(t) = (x(t), y(t), 0))$;
- $\boldsymbol{q}_3(t)$ moves on z-axis: $\boldsymbol{q}_3(t) = (0, 0, z(t))$.



3 Planar Sitnikov problem

We consider the planar Sitnikov problem which is the limiting case of the Sitnikov problem as the eccentricity of the Keplerian orbits goes to 1 :

- $\boldsymbol{q}_1(t)$ and $\boldsymbol{q}_2(t)$ move on the x-axis and satisfy the collinear Kepler problem $\ddot{x} = -2^{-3}x^{-2}$($\boldsymbol{q}_1(t) = -\boldsymbol{q}_2(t) = (x(t), 0)$) with collisions at $t \in \mathbb{Z}$(i. e. $x(t) = 0(t \in \mathbb{Z})$).
- $\boldsymbol{q}_3$ moves on y-axis:$\boldsymbol{q}_3(t) = (0, y(t))$;



4 Symbolic sequences

Define $\mathcal{M}$ by the set of bi-infinity sequences of $-1, 1$ such that the length of any successive part of -1 or 1 is no less than 3,

i.e., $\mathcal{M} \subset \{-1, 1\}^{\mathbb{Z}}$ and

for $\mathbf{a} = \{a_n\}_{n\in\mathbb{Z}} \in \{-1, 1\}^{\mathbb{Z}}$,

$\mathbf{a} \in \mathcal{M} \iff k_j \geq 3 (\forall j)$ where

$$\mathbf{a} : \ldots, \underbrace{-1, -1, \ldots, -1}_{k_{-1}}, \underbrace{1, 1, \ldots, 1}_{k_0}, \underbrace{-1, -1, \ldots, -1}_{k_1}, \underbrace{1, 1, \ldots, 1}_{k_2}, \ldots$$

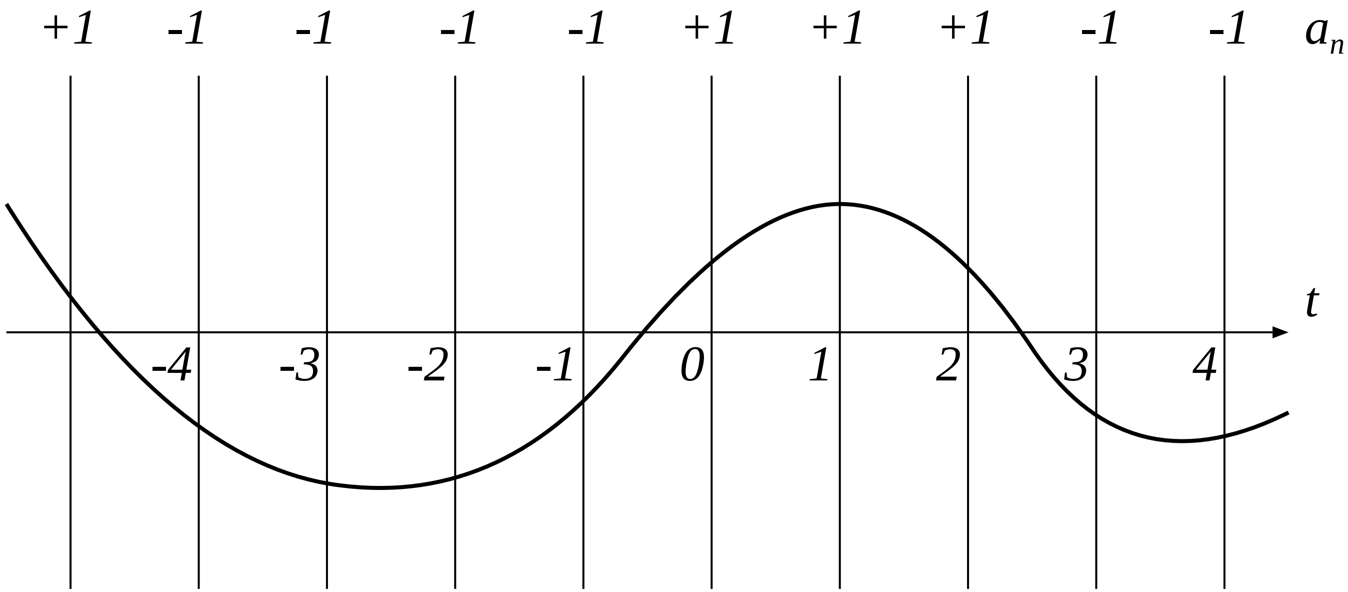
Let $\boldsymbol{e}_{\pm} \in \mathcal{M}$ be two trivial sequences:

$$\boldsymbol{e}_{+} = \ldots, 1, 1, 1, \ldots, \quad \boldsymbol{e}_{-} = \ldots, -1, -1, -1, \ldots$$

5 Sets of functions

For $\boldsymbol{a} \in \mathcal{M}$ and integers $N_1 < N_2$, define the sets of functions as follows:

$$\Omega_{N_1,N_2}(\mathbf{a}) = \{y(t) \in H^1([N_1, N_2], \mathbb{R}) \mid a_n \cdot y(n) > 0 \ (n = N_1, \ldots, N_2)\},$$
$$\Omega(\mathbf{a}) = \{y(t) \in H^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}) \mid a_n \cdot y(n) > 0 \ (n \in \mathbb{Z})\}.$$



6 Main Results

Consider the planar Sitnikov problem.

Theorem 1. *For any* $\boldsymbol{a} \in \mathcal{M}$ *and integers* $N_1 < N_2$, *there is a solution* $y(t) \in \Omega_{N_1,N_2}(\boldsymbol{a})$ *of the planar Sitnikov problem. Moreover* $a_{N_1}y(N_1), a_{N_2}y(N_2) > 0$ *can be arbitrarily prescribed.*

For $N \in \mathbb{N}$, define the set of the periodic sequences: $\mathcal{P}_N = \{\boldsymbol{a} \in \mathcal{M} \backslash \{\boldsymbol{e}_\pm\} \mid a_n = a_{n+N} (\forall n)\}$.

Theorem 2. *For any* $\boldsymbol{a} \in \mathcal{P}_N$, *there is a* N*-periodic solution* $y(t) \in \Omega(\boldsymbol{a})$ *of the planar Sitnikov problem.*

6 Main Results

Theorem 3. *Take* $\boldsymbol{a} = \{a_n\} \in \mathcal{M}$ *and* $\boldsymbol{b}^{\pm} = \{b_n^{\pm}\} \in \mathcal{P}_N$*. Assume that there are* $I_1 < I_2$ *such that* $a_n = b_n^-$ *for* $n < I_1$ *and* $a_n = b_n^+$ *for* $n > I_2$*, and that there is* $l \in \mathbb{Z}$ *such that at least one of the following satisfies*

1. $b_n^+ = b_{n+l}^-$ *for any* $n \in \mathbb{Z}$*;*
2. $b_n^+ = b_{-n+l}^-$ *for any* $n \in \mathbb{Z}$*;*
3. $b_n^+ = -b_{n+l}^-$ *for any* $n \in \mathbb{Z}$*;*
4. $b_n^+ = -b_{-n+l}^-$ *for any* $n \in \mathbb{Z}$*.*

Then there are periodic orbits $p^{\pm}(t) \in \Omega(\boldsymbol{b}^{\pm})$ *corresponding to* $\boldsymbol{b}^{\pm}$ *and a connecting orbit* $y(t) \in \Omega(\boldsymbol{a})$ *from* $p^-(t)$ *to* $p^+(t)$*.*

7 Proof (Existence of Minimizers)

The planar Sitnikov problem is equivalent to the variational problem with respect to the action functional

$$\mathcal{A}_{T_1,T_2}(y) = \int_{T_1}^{T_2} \frac{1}{2}\dot{y}^2 + \frac{1}{\sqrt{y^2 + x(t)^2}} dt.$$

[Theorem 1] Fix any $c, d > 0$. There is a minimizer $y_* \in \overline{\Omega_{N_1,N_2}(\boldsymbol{a})}$ of $\mathcal{A}_{N_1,N_2}$ on

$$\{y \in \Omega_{N_1,N_2}(\boldsymbol{a}) \mid y(N_1) = a_{N_1}c, y(N_2) = a_{N_2}d\}.$$

[Theorem 2] There is a minimizer $y_* \in \overline{\Omega_{0,N}(\boldsymbol{a})}$ of $\mathcal{A}_{0,N}$ on

$$\{y \in \Omega_{0,N}(\boldsymbol{a}) \mid y(0) = y(N)\}.$$

6 Proof (Minimization)

[Theorem 3] Let

$$\alpha_{\pm} = \inf_{y \in \Omega_{0,N}(\boldsymbol{b}_{\pm})} \mathcal{A}_{0,N}(y).$$

From the assumption, we get $\alpha_+ = \alpha_- =: \alpha$.

For $y \in \Omega(\boldsymbol{a})$, define

$$\mathcal{B}(y) = \sum_{k=-\infty}^{\infty} \left(\mathcal{A}_{k,k+1}(y) - \frac{\alpha}{N} \right).$$

We can show the existence of a minimizer $y_* \in \overline{\Omega(\boldsymbol{a})}$ of $\mathcal{B}$ by using Rabinowitz's result(1994).

7 Proof (Elimination of collisions)

We prove that y_* has no collision. Assume that $a_0 = -1$ and $y(0) = 0$.

From Sundmann's estimate, we get

$$x(t) = ct^{2/3} + O(t), y(t) = d_\pm t^{2/3} + O(t),$$

where $c = 2^{-4/3} \cdot 3^{2/3}, d_\pm \in \{-2^{-1} \cdot 3^{7/6}, 0, 2^{-1} \cdot 3^{7/6}\}$.

We make a modified curve with less value of the functional.

Define

$$\delta_\varepsilon = \begin{cases} -\varepsilon^{-1}(t+\varepsilon^2) - \varepsilon & (t \in [-2\varepsilon^2, -\varepsilon^2]) \\ -\varepsilon & (t \in [-\varepsilon^2, \varepsilon^2]) \\ \varepsilon^{-1}(t-\varepsilon^2) - \varepsilon & (t \in [\varepsilon^2, 2\varepsilon^2]) \end{cases}$$

and estimate

$$\mathcal{A}_{-1,1}(y+\delta_\varepsilon) - \mathcal{A}_{-1,1}(y).$$

7 Proof (Elimination of collisions)

We can estimate as follows:

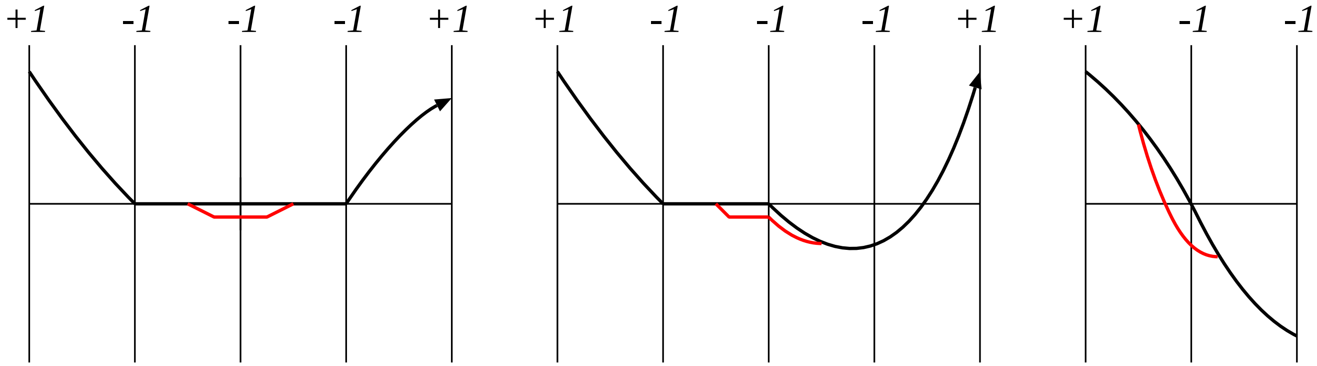
$$\mathcal{A}_{-1,1}(y+\delta_\varepsilon) - \mathcal{A}_{-1,1}(y) = -\frac{6}{(c^2+d_+^2)^{1/2}}\varepsilon^{2/3} + O(\varepsilon).$$

It is difficult in the cases of

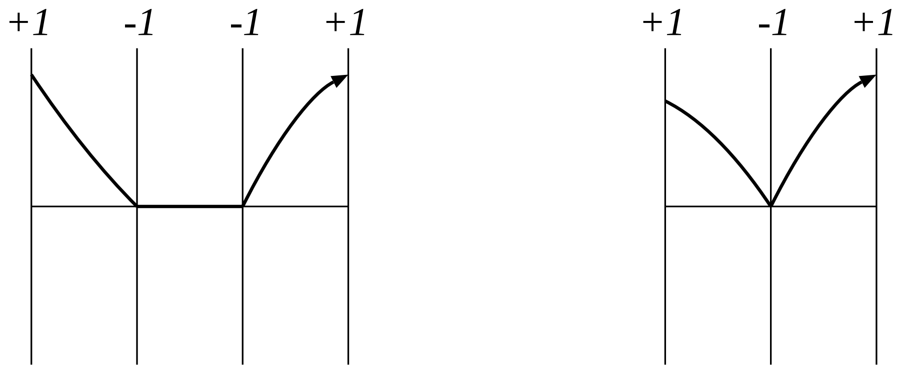
- $d_- = d_+ = 2^{-1} \cdot 3^{7/6}$;
- $d_- = +2^{-1} \cdot 3^{7/6}, d_+ = 0$;
- $d_- = 0, d_+ = +2^{-1} \cdot 3^{7/6}$.

7 Proof (Elimination of collisions)

We can eliminate collisions in the case of the following figures:



It is difficult to eliminate collisions in the case of the following figures:



Thank you.