On non-convergence of equilibrium measures at zero temperature limit

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Joint work with J.-R. Chazottes

Zero temperature limit problem

- What happens when a system is cooling down ?

$$\mathcal{P}(\beta\phi) = \sup\{h_\mu + \beta \int \phi \, d\mu : \mu \in \mathcal{M}_\sigma\} \quad \text{where} \quad \beta = \frac{1}{\text{temperature}}$$

letting temperature go to zero corresponds to $\beta \to \infty$

1. Setting
2. Convergence and non-convergence
3. Basic properties of zero temperature limits
4. Idea of the construction

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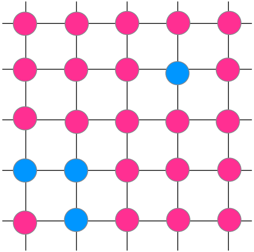
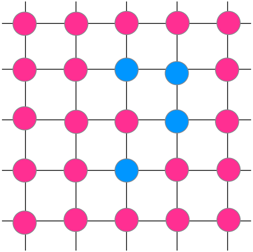
   - $\mathbb{Z}^d$ action
   - Zero temperature limit
2. Convergence and non-convergence
   - One-dimensional case
   - Multi-dimensional case
3. Basic properties of zero temperature limits
   - Maximizing measures
   - Maximal entropy measures
4. Idea of the construction
   - Subshifts
   - Construction of an effective subshift
   - Imbedding into a SFT

$\mathbb{Z}^d$ action

- Σ finite set (e.g. $\{pink, blue\}$)

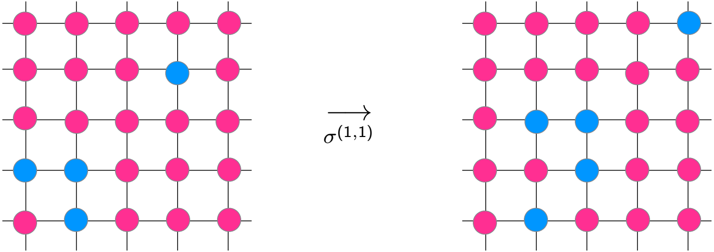
- $\Sigma^{\mathbb{Z}^d}$ where $d \geq 1$
- $d(x, y) = 2^{-N}$ where $N = \min\{\max\{n_i : i = 1, \ldots, d\} : x_n \neq y_n\}$





$\mathbb{Z}^d$ action

- Σ finite set
- $\Sigma^{\mathbb{Z}^d}$ where $d \geq 1$
- $d(x, y) = 2^{-N}$ where $N = \min\{\max\{n_i : i = 1, \ldots, d\} : x_n \neq y_n\}$
- $\Sigma^{\mathbb{Z}^d} \curvearrowleft^{\sigma} \mathbb{Z}^d$ by translation, i.e.
  $\{\sigma^m(x)\}_n = x_{n+m}$ for all $n, m \in \mathbb{Z}^d, x \in \Sigma^{\mathbb{Z}^d}$



Thermodynamic Formalism for $\mathbb{Z}^d$ action

- $\mathcal{M}_\sigma$ the set of σ-invariant Borel probability measures
  i.e. $\mu \in \mathcal{M}_\sigma$ if $\mu(\sigma^{-m}B) = \mu(B)$ for all $m \in \mathbb{Z}^d$ and Borel sets B.
- For a continuous function ϕ define its pressure $\mathcal{P}(\phi)$ by
  $\mathcal{P}(\phi) = \sup\{h_\mu + \int \phi \, d\mu : \mu \in \mathcal{M}_\sigma\}$
- $\mu \in \mathcal{M}_\sigma$ is an equilibrium measure for ϕ if $\mathcal{P}(\phi) = h_\mu + \int \phi \, d\mu$

Remarks:

- For a continuous function there always exists an equilibrium measure
- In the case of $d \geq 2$ the uniqueness is not always true even for locally constant functions

Zero temperature limit

- $\phi : \Sigma^{\mathbb{Z}^d} \to \mathbb{R}$ continuous function
- β inverse temperature parameter
- $\{\mu_\beta\}$ a sequence of equilibrium measures for $\beta\phi$

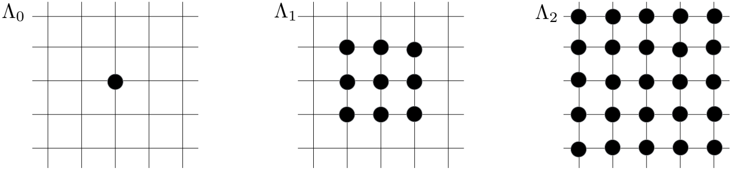
Problem 1

Does the limit $\lim_{\beta\to\infty} \mu_\beta$ *exist?*

Remark: We consider continuous parameter families, since there always exists a convergent subsequence by compactness of $\mathcal{M}_\sigma$

Regularity of constant functions

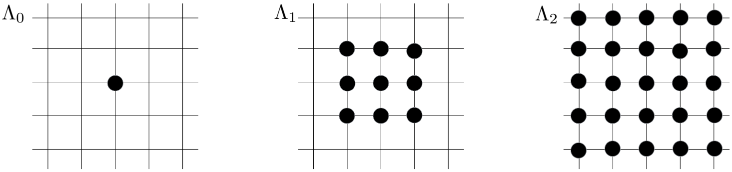
- $\Lambda_n = \{-n, -n+1, \ldots, n-1, n\}^d$ where $n \geq 0$.
- $\phi : \Sigma^{\mathbb{Z}^d} \to \mathbb{R}$ is locally constant if
  there exists $n \geq 0$ such that $\phi(x) = \phi(y)$ whenever $x|_{\Lambda_n} = y|_{\Lambda_n}$



Regularity of constant functions

- $\phi : \Sigma^{\mathbb{Z}^d} \to \mathbb{R}$ is locally constant if
  there exists $n \geq 0$ such that $\phi(x) = \phi(y)$ whenever $x|_{\Lambda_n} = y|_{\Lambda_n}$

- $\phi : \Sigma^{\mathbb{Z}^d} \to \mathbb{R}$ is Lipschitz continuous if
  there exists a constant $L > 0$ s.t. $|\phi(x) - \phi(y)| \leq Ld(x, y)$
  - $d(x, y) = 2^{-N}$ where $N = \min\{\max\{n_i : i = 1, \ldots, d\} : x_n \neq y_n\}$



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In the case of $d = 1$

- For locally constant functions there always exists the limit. (Bremont, Leplaideur, Chazottes et. al)
- There exists a Lipschitz continuous function such that the limit does not exist. (Chazottes and Hochman, Coronel and Letelier)

Remark 2

There is a unique choice of the sequence of equilibrium measures.

In the case of $d \geq 2$

- $d \geq 3$: there exists a locally constant function such that the limit does not exists for all sequence of equilibrium measures. (Chazottes and Hochman)
- $d \geq 2$: there exists a Lipschitz continuous function such that the limit does not exists for all sequence of equilibrium measures. (Coronel and Letelier)

Main Theorem

In dimension 2 there exists a locally constant function such that the limit does not exists for all sequence of equilibrium measures.

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First criterion: maximizing measures

- Any accumulation points of $\{\mu_\beta\}$ are maximizing measures for ϕ,

  i.e. $\int \phi \, d\mu_{max} = \sup\{\int \phi \, d\nu : \nu \in \mathcal{M}_\sigma\}$

$$\frac{1}{\beta}\sup\{h_\mu + \beta \int \phi \, d\mu : \mu \in \mathcal{M}_\sigma\} = \frac{1}{\beta}\left(h_{\mu_\beta} + \beta \int \phi \, d\mu_\beta\right)$$

This criterion implies that
the limit exists if there exists an unique maximizing measure for ϕ.

- $X(\phi)$ the maximizing subshift

  i.e. a subshift contains supports of all ϕ-maximizing measures.

Second criterion: maximal entropy measures

- If the limit exits, the entropy of the limit measure should be maximal among ϕ-maximizing measures

  i.e. $h_{\mu_\infty} = \sup\{h_\nu : \nu \in \mathcal{M}_\sigma(\phi)\}$

This criterion implies that
the limit exists if there exists an unique maximal entropy measure of $X(\phi)$.

Main Theorem

Main Theorem

In dimension 2 there exists a locally constant function s.t.

the limit does not exists for all sequence of equilibrium measures.

According to the criterions, we consider

- a subshift with more than two ergodic maximal entropy measures
- a function which is constant on the subshift

in order to get a non-convergence example.

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Subshifts

- Σ alphabet set
- $X \subset \Sigma^{\mathbb{Z}^d}$ is a subshift if it is σ-invariant and closed
- F forbidden set ($\subset \cup_{n\geq 1}\Sigma^{\Lambda_n}$)

$$X_F = \left\{ x \in \Sigma^{\mathbb{Z}^d} : \text{no pattern from } F \text{ appears in } x \right\}$$

  - X_F is σ-invariant and closed

- X is a SFT if
  there exists a finite set F s.t. $X = X_F$
- X is effective if
  there exists a recursively enumerable set F s.t. $X = X_F$

Ideas of the constructions

1. Define an effective subshift $X \subset \Sigma^{\mathbb{Z}}$
2. Imbed X into a subshift of finite type (SFT) $Y' \subset \mathcal{B}'^{\mathbb{Z}^2}$
3. Consider the function ϕ_F defined by the forbidden set F of Y' i.e.,

$$\phi_F(x) = \begin{cases} -1 & \text{if} \quad x|_{\Lambda_F} \in F \\ 0 & \text{else} \end{cases}$$

where Λ_F is the "shape" of the forbidden patterns.

Ideas of the constructions

$$\phi_F(x) = \begin{cases} -1 & \text{if} \quad x|_{\Lambda_F} \in F \\ 0 & \text{else} \end{cases}$$

Note that

- ϕ_F is locally constant
- Y' is the maximizing subshift of ϕ_F
  - $Y' = \left\{ y \in \mathcal{B}'^{\mathbb{Z}^2} : \text{no pattern from } F \text{ appears in } y \right\}$

Ideas of the constructions

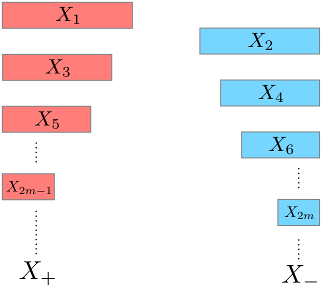
1. Define an effective subshift $X \subset \Sigma^{\mathbb{Z}}$
   - sequence of SFTs
2. Imbed X into a SFT $Y' \subset {\mathcal{B}'}^{\mathbb{Z}^2}$
   - simulation of a Turing Machine by a self-similar tile set
3. Consider the function ϕ_F defined by the forbidden set F of Y' i.e.,

$$\phi_F(x) = \begin{cases} -1 & \text{if} \quad x|_{\Lambda_F} \in F \\ 0 & \text{else} \end{cases}$$

where Λ_F is the "shape" of the forbidden patterns.

Construction of X

- $\Sigma = \{-1, 0, +1\}$ alphabet set $\qquad \Sigma_{\pm} = \{0, \pm 1\}$
- $\{F_k\}$ sequence of forbidden sets
- $X_k := X_{F_k}$ for all $k \geq 1$



$X = X_+ \cup X_-$

$h(X_1) > h(X_2) > h(X_3) > \cdots$

Ideas of the constructions

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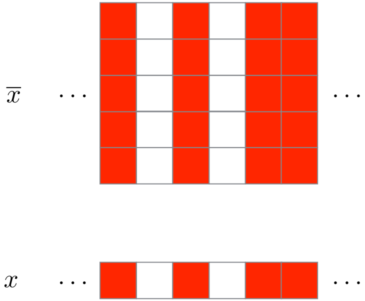
Imbedding into two-dimensional SFT

- For $x \in \Sigma^{\mathbb{Z}}$ denote by $\overline{x}$ its vertical extension i.e.,

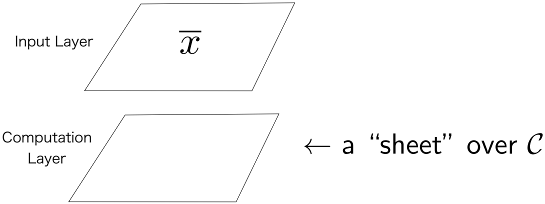
$$\overline{x}_{i,j} = x_i \quad \text{for all } i,j \in \mathbb{Z}$$



Let $\overline{X} = \{\overline{x} : x \in X\}$

Imbedding into two-dimensional SFT

Theorem 3 (B. Durand, A. Romashchenko and A. Shen (2012))
Let X be an effective subshift over Σ.
Then there exist an alphabet set $\mathcal{B} \subset \Sigma \times \mathcal{C}$, a map $r : \mathcal{B} \to \Sigma$ and a two-dimensional SFT Y over $\mathcal{B}$ s.t. $r(Y) = \overline{X}$.



The program to check the forbidden strings of X is simulated on the computation layer

- X the effective subsfhit
- $\overline{X}$ the vertical extension of X
- Y the two-dimensional SFT whose input layers are in $\overline{X}$
- Y' the two-dimensional SFT with a (finite) forbidden set F
- ϕ_F the locally constant function defined by F

the limit does not exists for all sequence of equilibrium measures of ϕ_F !

Further questions

Main Theorem

There exist an alphabet set $\mathcal{B}'$ and a SFT Y' over $\mathcal{B}'$ s.t. the limit does not exists for all sequence of equilibrium measures of the locally constant function ϕ_F defined by the forbidden set for Y'

- The size of our alphabet set is very large.
  - Can we reduce the number of alphabets?
  - Is there a critical size of alphabet sets for which every locally constant function converges?