Applications of the Helmholtz-Hodge decomposition to the study of vector fields

Tomoharu Suda

Graduate School of Human and Environmental Studies
Kyoto University

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Outline

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Introduction

The Helmholtz–Hodge decomposition (HHD) is a decomposition of vector fields whereby they are expressed as the sum of a gradient vector field and a divergence-free vector field.
Although there are examples of HHD applied to the detection of singularities of vector fields, few studies have been concerned with HHD from the viewpoint of dynamical systems.

Notation

It is assumed that all vector fields are C^2. We consider autonomous differential equations given in the form

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}), \tag{1}$$

where $\mathbf{F}(\mathbf{x})$ is a smooth vector field defined on $\bar{\Omega}$, where $\Omega \subset \mathbb{R}^n$ is an open domain which contains the origin. In what follows, it is assumed that the domain Ω is bounded and has C^2-boundary.

Notation

Definition (Derivative along solutions)

For a C^1-function f on a domain $\Omega \subset \mathbb{R}^n$, its **derivative along solutions** $\dot{f}$ is defined by

$$\dot{f}(\mathbf{x}_0) := \frac{d}{dt} f\left(\mathbf{x}(t)\right)\Big|_{t=0} = \sum_i \frac{\partial f}{\partial x_i} \frac{dx_i(t)}{dt}\Big|_{t=0},$$

where $\mathbf{x}(t)$ is a solution of the vector field with initial value $\mathbf{x}_0$. Here the vector field does not depend on t, and therefore $\dot{f}$ is given explicitly as a function depending only on $\mathbf{x}$, namely,

$$\dot{f}(\mathbf{x}_0) = \nabla f(\mathbf{x}_0) \cdot \mathbf{F}(\mathbf{x}_0).$$

Notation

We also use the following concept later.

Definition (Singular values of a matrix)

Let A be a square real matrix of order n. Then, the matrix ${}^{t}AA$ is nonnegative definite; therefore, it has positive eigenvalues. The square root of an eigenvalue of ${}^{t}AA$ is called a **singular value** of A.

Definition of HHD

The HHD is a decomposition of vector fields whereby they are expressed as the sum of a gradient vector field and a divergence-free vector field.

Definition (Helmholtz–Hodge decomposition)

For a vector field $\mathbf{F}$ on a domain $\Omega \subset \mathbb{R}^n$, its **Helmholtz–Hodge decomposition** (**HHD**) is a decomposition of the form

$$\mathbf{F} = -\nabla V + \mathbf{u},$$

where $V : \Omega \to \mathbb{R}$ is a C^3 function and $\mathbf{u}$ is a vector field on Ω with the property $\nabla \cdot \mathbf{u} = 0$. V is called a **potential function**.

Example

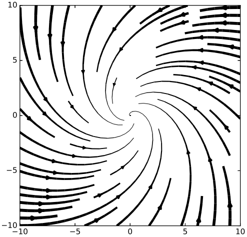

Figure: Solution curves of the vector field.

Let us consider the following differential equation:

$$\mathbf{x}' = \mathbf{F} = \begin{pmatrix} -x - y \\ -y + x \end{pmatrix}. \quad (2)$$

An HHD of the vector field is given by

$$V(x,y) := \frac{1}{2}x^2 + \frac{1}{2}y^2,$$

$$\mathbf{u}(x,y) := \begin{pmatrix} -y \\ x \end{pmatrix}.$$

Basic Properties of HHD

- Solutions to the Poisson equation $\Delta V = -\nabla \cdot \mathbf{F}$ gives the potential functions. In particular, if $\mathbf{F}$ is a polynomial vector field, there is a polynomial solution.
- HHD is not unique. Addition of harmonic function yields another choice of decomposition. That is, if $\mathbf{F} = -\nabla V + \mathbf{u}$ is an HHD and h is a harmonic function, we have another HHD $\mathbf{F} = -\nabla(V + h) + (\mathbf{u} + \nabla h)$.
- HHD does not necessarily respect the behavior of vector fields. For example, it is not always true that $\mathbf{F} = \mathbf{0}$ implies $\mathbf{u} = \nabla V = \mathbf{0}$.

Choice of Decomposition

Since HHD is not unique and does not necessarily respect the dynamics, we need additional conditions to obtain a 'useful' decomposition.
Several methods are proposed to obtain HHD that yields a good picture of dynamics. For example,

- Requiring orthogonality in L^2–sense. This is usually achieved via imposing $\mathbf{u} \cdot \mathbf{n} = 0$, where $\mathbf{n}$ is normal vector of the domain considered.
- Construction via Green function.

However, they do not necessarily respect the behavior of vector fields and the justification is not clear.

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Definition of strictly orthogonal HHD

In certain cases, a strictly orthogonal HHD may be obtained, which is sufficient for completely describing the behavior of the vector field.

Definition

HHD $\mathbf{F} = -\nabla V + \mathbf{u}$ is said to be **strictly orthogonal** on $D \subset \mathbb{R}^n$ if $\mathbf{u}(\mathbf{x}) \cdot \nabla V(\mathbf{x}) = 0$ for all $\mathbf{x} \in D$.

Remarks.

- Gradient or divergence-free vector fields are examples of vector fields with a strictly orthogonal HHD.
- For a strictly orthogonal HHD, $\mathbf{F} = \mathbf{0}$ implies $\mathbf{u} = \nabla V = \mathbf{0}$.

In general, vector fields may not have a strictly orthogonal HHD. Nor is it necessarily unique. However, we have the following results.

Theorem

Let A be a normal matrix. Then, the vector field $\mathbf{F} = A\mathbf{x}$ has a strictly orthogonal HHD.

Theorem

Let $\mathbf{F}$ be a vector field defined on $\mathbb{R}^2$. Let $D \subset \mathbb{R}^2$ be a domain and ∂D be a partially C^1 Jordan closed curve. It is assumed that there is a point $\mathbf{x} \in D$ such that $\omega(\mathbf{x}) = \partial D$. Then, a strictly orthogonal HHD is unique if it exists.

Example

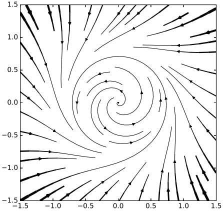
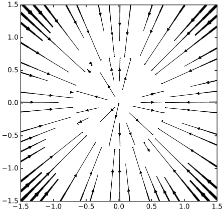


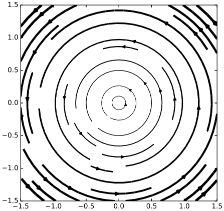
Figure: Solution curves of the vector field.

For $\mu > 0$ and $\omega < 0$, the following differential equation is considered:

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \mu x - \omega y - (x^2 + y^2)\,x \\ \omega x + \mu y - (x^2 + y^2)\,y \end{pmatrix}. \tag{3}$$

Example





This vector field has a strictly orthogonal HHD, namely,

$$V = -\frac{\mu}{2}(x^2+y^2)+\frac{1}{4}(x^2+y^2)^2,$$

$$\mathbf{u} = \begin{pmatrix} -\omega y \\ \omega x \end{pmatrix}.$$

Upper: solution curves of $-\nabla V$.
Lower: solution curves of $\mathbf{u}$.

Properties of Strictly Orthogonal HHD

Vector fields with strictly orthogonal HHD are a generalization of gradient vector fields.

Theorem

Let $D \subset \mathbb{R}^n$ be a bounded domain. If a vector field $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ has a strictly orthogonal HHD on $\bar{D}$ with potential function V, then the following hold:

1. *If D is forward invariant, then for all $\mathbf{x} \in D$,*

$$\omega(\mathbf{x}) \subset \{\mathbf{y} \in \bar{D} \,|\, \nabla V(\mathbf{y}) = \mathbf{0}\}.$$

2. *If D is backward invariant, then for all $\mathbf{x} \in D$,*

$$\alpha(\mathbf{x}) \subset \{\mathbf{y} \in \bar{D} \,|\, \nabla V(\mathbf{y}) = \mathbf{0}\}.$$

Relationship with other conditions

In some cases, strictly orthogonal HHD coincides with that obtained by other conditions often used. For example, the following result holds.

Lemma

Let $\mathbf{F}$ be a vector field defined on $\mathbb{R}^2$. Let $D \subset \mathbb{R}^2$ be a domain and ∂D be a piecewise C^1 Jordan curve. It is assumed that there is a point $\mathbf{x} \in D$ such that $\omega(\mathbf{x}) = \partial D$. If there is a strictly orthogonal HHD of $\mathbf{F}$ on $\bar{D}$, it satisfies the following on ∂D:

$$\mathbf{u} \cdot \mathbf{n} = 0,$$

where $\mathbf{n}$ is normal vector of ∂D pointing outward.

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Definition of Lyapunov functions

Definition (Lyapunov functions)

Let $\mathbf{x}_*$ be an equilibrium point and $U \subset \mathbb{R}^n$ a neighborhood of $\mathbf{x}_*$. A C^1-function $L : U \to \mathbb{R}$ is said to be a **(local) Lyapunov function** if the following hold:

1. For all $\mathbf{x} \in U - \{\mathbf{x}_*\}$, we have $L(\mathbf{x}) > L(\mathbf{x}_*)$.
2. On the set $U - \{\mathbf{x}_*\}$, we have $\dot{L} \leq 0$.

Construction of Lyapunov functions using HHD

Using HHD, we can construct a Lyapunov function near equilibrium if a stability condition is satisfied.
First we ensure that the potential function assumes a minimum value at the equilibrium point.

Theorem

Let $\mathbf{F} : \bar{\Omega} \to \mathbb{R}^n$ *be a vector field with an equilibrium point at the origin. If* $\nabla \cdot \mathbf{F}(\mathbf{0}) < 0$*, then it has an HHD such that the potential function* V *has a minimum at the origin.*

Construction of Lyapunov functions using HHD

The next theorem provides a sufficient condition whereby the potential function is a Lyapunov function.

Theorem

Let $\mathbf{F} : \bar{\Omega} \to \mathbb{R}^n$ *be a vector field with an equilibrium point at the origin and* $\mathbf{F} = -\nabla V + \mathbf{u}$ *be an HHD, assume* V *has a minimum at the origin, and let the largest singular value of* $D\mathbf{u}_\mathbf{0}$ *be* $\lambda_\mathbf{u}$*, the smallest singular value of* $D\mathbf{F}_\mathbf{0}$ *be* $\mu_\mathbf{F}$*, and the smallest eigenvalue of* $\mathrm{Hess}\, V$ *be* μ_V*. If* $\lambda_\mathbf{u}^2 - \mu_V^2 < \mu_\mathbf{F}^2$*, then* V *is a (local) Lyapunov function of the origin.*

Example

Let the vector field **F** on $\mathbb{R}^2$ be given by

$$\mathbf{F}(x, y) = \begin{pmatrix} -a & b \\ c & -b \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

with $a > b > c > 0$. Then, there is an HHD of $\mathbf{F} = -\nabla V + \mathbf{u}$ given by

$$\begin{aligned} V &= \frac{a}{2}x^2 + \frac{b}{2}y^2, \\ \mathbf{u} &= \begin{pmatrix} by \\ cx \end{pmatrix}. \end{aligned}$$

As $\lambda_{\mathbf{u}} = \mu_V = b$, it suffices to show that $\mu_{\mathbf{F}}^2 > 0$. This is equivalent to the matrix ${}^t(D\mathbf{F_0})(D\mathbf{F_0})$ being regular, which is obviously true.

Example

This analysis may be extended to vector fields involving higher-order terms with the same linear part.
For example, let a vector field **F** be given by

$$\mathbf{F} = \begin{pmatrix} -a & b \\ c & -b \end{pmatrix} \mathbf{x} - \nabla p + \begin{pmatrix} q \\ r \end{pmatrix},$$

where p, q, r are homogeneous polynomials with $\deg(p) > 2$, $\deg(q), \deg(r) > 1$, and $\frac{\partial q}{\partial x} + \frac{\partial r}{\partial y} = 0$. Then,
$V = \frac{a}{2}x^2 + \frac{b}{2}y^2 + p(x, y)$ is a Lyapunov function of the origin.

Summary

- The definition and basic properties of HHD are reviewed.
- Strictly orthogonal HHD enables a detailed analysis of vector fields, if it exists.
- HHD can also be applied to the construction of Lyapunov functions.