# LARGE DEVIATION PRINCIPLE FOR ARITHMETIC FUNCTIONS IN CONTINUED FRACTION EXPANSION

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Abstract. Khinchin proved that the arithmetic mean of continued fraction digits of Lebesgue almost every irrational number in $(0,1)$ diverges to infinity. Hence, none of the classical limit theorems such as the weak and strong laws of large numbers or central limit theorems hold. Nevertheless, we prove the existence of a large deviations rate function which estimates exponential probabilities with which the arithmetic mean of digits stays away from infinity. This leads us to a contradiction to the widely-shared view that the Large Deviation Principle is a refinement of laws of large numbers: the former can be more universal than the latter.

## 1. Introduction

Each irrational number $x \in (0,1)$ has the continued fraction expansion

$$x = \cfrac{1}{a_1(x) + \cfrac{1}{a_2(x) + \cdots}}.$$

The statistics of the continued fraction digits $a_1, a_2, \ldots$ have been studied at least since the time of Carl Friedrich Gauss. The following is a consequence of Khinchin's formula [21, Theorem 35] and Birkhoff's ergodic theorem: Let $\psi\colon \mathbb{N}\setminus\{0\} \to \mathbb{R}$ be a non-negative function for which there exist $c>0$ and $\rho>0$ such that for every $n \in \mathbb{N}\setminus\{0\}$, $\psi(n) < cn^{1-\rho}$. Denote by $\lambda$ the restriction of the Lebesgue measure to $(0,1)$. Then

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\psi(a_k) = \sum_{n=1}^{\infty}\frac{\psi(n)}{\log 2}\log\left(1+\frac{1}{n(n+2)}\right) \quad \lambda\text{-a.e.} \tag{1.1}$$

Taking $\psi(n) = \log n$ yields

$$\lim_{n\to\infty}\sqrt[n]{a_1a_2\cdots a_n} = K \quad \lambda\text{-a.e.},$$

where $K = 2.6854...$ is Khinchin's constant. In particular,

$$\liminf_{n\to\infty}\frac{a_n}{n} = 0 \quad \lambda\text{-a.e.}$$

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Since $\sum_{n=1}^{\infty} n\frac{1}{n(n+1)} = \infty$, (1.1) is not valid for $\psi(n) = n$. Khinchin [21] noted that for $\lambda$-almost every $x \in (0,1)$ the inequality $a_n(x) > n \log n$ holds for infinitely many $n \in \mathbb{N}$ as a consequence of Borel-Bernstein's theorem [1, 2, 3], and thus

$$\limsup_{n\to\infty} \frac{a_n}{n} = \infty \quad \lambda\text{-a.e.}$$

and $S_n/n$ does not converge $\lambda$-a.e. where $S_n = a_1 + a_2 + \cdots + a_n$. In fact,

$$\lim_{n\to\infty} \frac{S_n}{n} = \infty \quad \lambda\text{-a.e.} \tag{1.2}$$

Hence, none of the classical limit theorems in probability, such as the weak and strong laws of large numbers and central limit theorems hold for the sum. Philipp [26] strengthened (1.2) by showing that, for a sequence $\theta(n)$ of positive numbers for which $\theta(n)/n$ is non-decreasing,

$$\lim_{n\to\infty} \frac{S_n}{\theta(n)} = 0 \quad \text{or} \quad \limsup_{n\to\infty} \frac{S_n}{\theta(n)} = \infty \quad \lambda\text{-a.e.},$$

according as the series $\sum_{n=1}^{\infty} 1/\theta(n)$ converges or not. This intricacy of the stochastic property of the sum is due to the occurrence of rate but exceptionally large digits. Diamond and Vaaler [8] showed that if the largest digit in $a_1 + \cdots + a_n$ is trimmed then the strong law of large numbers holds with norming constants $n \log n$. Philipp [26] showed that the sum satisfies a central limit theorem if a few of the largest digits are trimmed. Distributional limit theorems for the sum were obtained in [13, 14]. Kesseböhmer and Slassi [18, 19] introduced stopping times and established several limit theorems on fluctuations of the sum.

In view of the results of Khinchin and Philipp, much attention has been given to determining the Hausdorff dimension of exceptional sets

$$\left\{ x \in (0,1) \colon \lim_{n\to\infty} \frac{S_n(x)}{\theta(n)} = \alpha \right\},$$

where $\alpha \in \mathbb{R}$ is a constant. See e.g., [5, 10, 15] with $\theta(n) = n$ and [23, 31, 32] with $\theta(n)$ growing faster than $n$. From the viewpoint of large deviations, it is also relevant to consider the following set

$$\left\{ x \in (0,1) \colon \frac{S_n(x)}{n} \leq \alpha \right\},$$

where $\alpha \in \mathbb{R}$ is a constant. (1.2) implies that, for every $\alpha \in \mathbb{R}$ the Lebesgue measure of this set goes to 0 as $n \to \infty$. In this paper we show that this convergence is exponential. More precisely, we establish the (level-1) Large Deviation Principle (LDP for short), i.e., show the existence of a rate function which estimates exponential probabilities with which $S_n/n$ stays away from $\infty$.

**Main Theorem.** *Let* $\psi \colon \mathbb{N}\backslash\{0\} \to \mathbb{R}$ *satisfy* $\liminf_{n\to\infty} \psi(n) = \infty$ *and* $\int \psi(a_1(x))dx = \infty$. *There exists* $\alpha^- \in \mathbb{R}$ *such that the following holds:*

- *for every $\alpha \in \mathbb{R}$ the limit*

$$J(\alpha) := -\lim_{n\to\infty}\frac{1}{n}\log\lambda\left\{x\in(0,1)\colon \frac{1}{n}\sum_{k=1}^{n}\psi(a_k(x))\leq\alpha\right\}$$

*exists and is finite if and only if $\alpha \geq \alpha^-$. The function $\alpha \in [\alpha^-,\infty) \mapsto J(\alpha)$ is lower semi-continuous, strictly positive, convex, strictly monotone decreasing and $J(\alpha)\to 0$ as $\alpha\to\infty$;*
- *for every $\alpha \in \mathbb{R}$,*

$$\lim_{n\to\infty}\frac{1}{n}\log\lambda\left\{x\in(0,1)\colon \frac{1}{n}\sum_{k=1}^{n}\psi(a_k(x))\geq\alpha\right\}=0.$$

Under the assumption of the Main Theorem the strong law of large numbers does not hold, namely $\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\psi(a_k)=\infty$ $\lambda$-a.e. This leads us to a contradiction to the widely-shared view that the LDP is a refinement of laws of large numbers: the former can be more universal than the latter. Notice that Donsker-Varadhan's formulation [9] of the LDP does not *a priori* assume laws of large numbers.

The continued fractions are generated by iterating *the Gauss map* $T\colon(0,1]\to(0,1]$ given by $T(x)=1/x-\lfloor 1/x\rfloor \pmod 1$. This map leaves invariant and ergodic the Borel probability measure $d\mu_T=\frac{1}{\log 2}\frac{dx}{1+x}$ that is absolutely continuous with respect to $\lambda$. The dynamics of $T$ is modeled by a topological Markov shift on a countably infinite number of alphabets. The proof of the Main Theorem is based on Theorem 1.1 below, on this symbolic dynamical system and associated arithmetic functions which are allowed to be unbounded.

We introduce our settings and terms in more precise terms. Denote by $X$ the set of all one-sided infinite sequences over $\mathbb{N}$ endowed with the product topology of the discrete topology on $\mathbb{N}$, namely

$$X=\{x=(x_0,x_1,\ldots)\colon x_i\in\mathbb{N},\ i\in\mathbb{N}\}.$$

Denote the left shift $\sigma\colon X\to X$ by $(\sigma(x))_i=x_{i+1}$, $i\in\mathbb{N}$. The continued fraction expansion is generated by iterating $T$, namely in the expansion (1), $a_i(x)=\lfloor 1/T^{i-1}(x)\rfloor$ and thus $a_i(x)=k$ if and only if $T^{i-1}(x)\in(\frac{1}{k+1},\frac{1}{k}]$. Following orbits of $T$ over the infinite Markov partition $\{(\frac{1}{k+1},\frac{1}{k}]\}_{k=1}^{\infty}$ of $(0,1]$ one can model $T$ by the left shift $\sigma$. The conjugacy between $T$ and $\sigma$ induces a one-to-one correspondence between $T$-invariant Borel probability measures and $\sigma$-invariant ones which preserves entropy and integrals of functions. To simplify notation, up to this conjugacy we identify measures invariant by the two systems and functions in the two spaces. In particular, this means that we allow expressions like $\int\log|DT|d\mu$ for a $\sigma$-invariant measure $\mu$.

Denote by $\mathcal{M}$ the space of Borel probability measures on $X$ endowed with the weak*-topology. As $X$ becomes a (non-compact) Polish space, the weak*-topology is metrizable and $\mathcal{M}$ becomes a Polish space. Denote by $\mathcal{M}(\sigma)$ the subspace of $\mathcal{M}$ consisting of $\sigma$-invariant ones. Write $\phi=-\log|DT|$ and set

$$\mathcal{M}_\phi(\sigma)=\left\{\mu\in\mathcal{M}(\sigma)\colon \int\phi d\mu>-\infty\right\}.$$

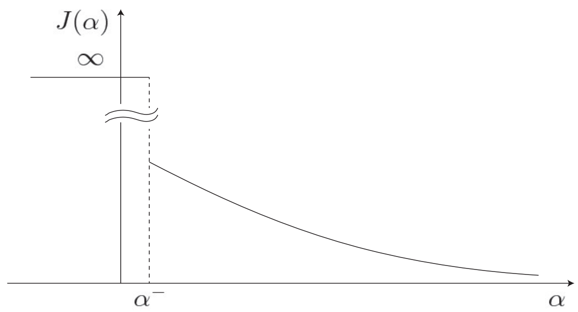


FIGURE 1. The graph of the function $J$ in the Main Theorem.

Denote by $\mathcal{M}_\phi^e(\sigma)$ the set of elements of $\mathcal{M}_\phi(\sigma)$ which are ergodic. For each $\mu \in \mathcal{M}(\sigma)$ denote by $h(\mu)$ the (Kolmogorov-Sinaĭ) entropy of $\mu$ with respect to $\sigma$, and define *the Lyapunov exponent* $\chi(\mu)$ of $\mu$ by $\chi(\mu) = -\int \phi d\mu$. Note that $\chi(\mu) \in [\frac{\sqrt{5}+1}{2}, \infty]$. If $\mu \in \mathcal{M}_\phi(\sigma)$ then $h(\mu) < \infty$ holds (see e.g. [24, Theorem 1.4]). For each $\mu \in \mathcal{M}_\phi(\sigma)$ write $F(\mu) = h(\mu) - \chi(\mu)$ and call it *the* (minus of the) *free energy*. It is known that $F \leq 0$ and $F(\mu) = 0$ holds if and only if $\mu = \mu_T$, see [25, 30]. For a function $\varphi\colon (0,1) \to \mathbb{R}$ and an integer $n \geq 1$ write $S_n\varphi = \sum_{i=0}^{n-1} \varphi \circ T^i$. Given $\psi\colon \mathbb{N} \setminus \{0\} \to \mathbb{R}$, for each integer $n \geq 1$ denote by $\lambda_n$ the distribution of $\frac{1}{n}S_n(\psi \circ a_1)$ with respect to $\lambda$. Note that $S_n(\psi \circ a_1) = \sum_{k=1}^n \psi(a_k)$. Put

$$\alpha^- = \inf\left\{\int \psi \circ a_1 d\mu\colon \mu \in \mathcal{M}_\phi(\sigma)\right\} \text{ and } \alpha^+ = \sup\left\{\int \psi \circ a_1 d\mu\colon \mu \in \mathcal{M}_\phi(\sigma)\right\}.$$

The infimum and the supremum are taken over all $\mu \in \mathcal{M}_\phi(\sigma)$ for which $\int \psi \circ a_1 d\mu$ is well-defined, including $\pm\infty$.

**Theorem 1.1.** (Level-1 Large Deviation Principle). *Let* $\psi\colon \mathbb{N} \setminus \{0\} \to \mathbb{R}$. *The function* $I\colon \mathbb{R} \to [0, \infty]$ *defined by*

$$I(\alpha) = \lim_{\epsilon \to 0} \inf\left\{-F(\mu)\colon \mu \in \mathcal{M}_\phi(\sigma), \left|\int \psi \circ a_1 d\mu - \alpha\right| < \epsilon\right\}$$

*is convex, lower semi-continuous and satisfies the following:*

- (lower bound) *for every open set* $U \subset \mathbb{R}$,
$$\liminf_{n\to\infty} \frac{1}{n} \log \lambda_n(U) \geq -\inf I|_U;$$
- (upper bound) *for every closed set* $C \subset \mathbb{R}$,
$$\limsup_{n\to\infty} \frac{1}{n} \log \lambda_n(C) \leq -\inf I|_C.$$

*In addition,* $I(\alpha) < \infty$ *if and only if* $\alpha^- \leq \alpha \leq \alpha^+$.

The function $I$ is called *a rate function.* Here and in what follows we follow the convention $\inf \emptyset = \infty$, $\sup \emptyset = -\infty$, $\log 0 = -\infty$. The novelty of Theorem 1.1 consists in the case where $\psi$ is unbounded. Otherwise, the level-1 LDP was already shown by Denker and Kabluchko [7, Theorem 3.3]. Our proof is a dynamical one inspired by the work of Takahashi [29], and gives an expression of the rate function in terms of free energies of invariant measures which is not apparent in [7]. This expression is essential for the proof of the Main Theorem.

Taking $\psi(n) = \log n$ in Theorem 1.1 yields the LDP for *the Khinchin exponent*. A close inspection of the proof of Theorem 1.1 shows that the arithmetic function $\psi \circ a_1$ may be replaced by another $\varphi\colon X \to \mathbb{R}$ for which $\sup_{w\in E^k}\sup_{x,y\in[w]} S_k\varphi(x) - S_k\varphi(y)$ is uniformly bounded in $k$. In particular, the LDP holds for the Lyapunov exponent of the Gauss map. The Khinchin and Lyapunov spectra of the Gauss map were determined in [11, 17, 27]. See also [12, 22].

The rest of this paper consists of two sections. In Sect.2 we finish the proof of the Main Theorem assuming the conclusion of Theorem 1.1. From Theorem 1.1 it follows that $J(\alpha) = I(\alpha)$, and the biggest difficulty is to show $I(\alpha) > 0$ for every $\alpha > \alpha^-$. We show that if $\alpha > \alpha^-$ and $I(\alpha) = 0$, then one would be able to find a convergent sequence to $\mu_T$ in $\mathcal{M}_\phi(\sigma)$ along which the (minus of the) free energy converges 0, which turns out to be absurd. One key assumption in deriving this contradiction is that $\psi$ blows up at infinity. In Sect.3 we prove Theorem 1.1.

## 2. On the proof of the Main Theorem

In this section we start with a few preliminary lemmas on sequences of measures in $\mathcal{M}_\phi(\sigma)$. Building on them and Theorem 1.1 we finish the proof of the Main Theorem.

### 2.1. **Tightness.**

As $X$ is non-compact, $\mathcal{M}$ is not weak*-compact. Hence the convergence of a sequence of probability measures is an issue. In order to establish the convergence we show the following tightness result.

**Lemma 2.1.** *Let* $\psi\colon \mathbb{N}\setminus\{0\} \to \mathbb{R}$ *satisfy* $\liminf_{n\to\infty}\psi(n) = \infty$. *Let* $\{\nu_n\}$ *be a sequence in* $\mathcal{M}(\sigma)$ *such that* $\sup_n \int \psi\circ a_1 d\nu_n < \infty$. *Then* $\{\nu_n\}$ *is tight.*

*Proof.* We modify the argument in the proof of [16, Lemma 2]. For an integer $M \geq 1$ put $X_M = \bigcup_{i=0}^{M-1}[i]$. Fix $M_0 \geq 0$ such that $\inf_{n\geq M_0}\psi(n) > 0$ holds. From the assumption in Lemma 2.1 there exists a constant $c > 0$ such that $\inf_{n\geq M}\psi(n)\sup_n \nu_n(X_M^c) < c$ holds for every $M \geq n_0$, namely

$$\sup_n \nu_n(X_M^c) < \frac{c}{\inf_{n\geq M}\psi(n)} \quad \text{for every } M \geq M_0. \tag{2.1}$$

Let $\epsilon > 0$. We construct an increasing sequence of positive integers $\{m_i\}_{i\geq 0}$ such that the compact set

$$A = \{x \in X\colon 0 \leq x_i \leq m_i \text{ for every } i \geq 0\}$$

satisfies $\nu_n(A) > 1-\epsilon$ for every $n$. Let $\pi_i\colon X\to\mathbb{N}$ be the projection onto the $i$-th coordinate. We have

$$\begin{aligned}\nu_n(A) &= \nu_n\left(X\cap\left(\bigcup_{i=0}^{\infty}\{x\in X\colon x_i>m_i\}\right)^c\right)\\ &= 1-\sum_{i=0}^{\infty}\nu_n\left(\{x\in X\colon x_i>m_i\}\right)\\ &= 1-\sum_{i=0}^{\infty}\nu_n(\pi_i^{-1}(X_{m_i+1}^c))\\ &= 1-\sum_{i=0}^{\infty}\nu_n(X_{m_i+1}^c)\\ &\geq 1-\sum_{i=0}^{\infty}\sup_n\nu_n(X_{m_i+1}^c),\end{aligned}$$

the last equality from the shift invariance of $\nu_n$. Therefore, in order to show the tightness of $\{\nu_n\}$ it is enough to find $\{m_i\}$ such that

$$\sup_n\nu_n(X_{m_i+1}^c)<\frac{\epsilon}{2^{i+1}}\quad\text{for every } i\geq 0.$$

This is possible by (2.1) and the assumption $\liminf_{n\to\infty}\psi(n)=\infty$. □

2.2. **Finiteness of Lyapunov exponent.** Having established the convergence of a sequence in $\mathcal{M}_\phi(\sigma)$ by Lemma 2.1, the next task is to show that the limit point belongs to $\mathcal{M}_\phi(\sigma)$. We will show this combining the next two lemmas. For a function $\varphi\colon X\to\mathbb{R}$ define

$$\mathrm{var}_1(\varphi)=\sup_{k\geq 0}\sup_{x,y\in[k]}\varphi(x)-\varphi(y).$$

**Lemma 2.2.** *Let $\varphi\colon X\to\mathbb{R}$ be a continuous function satisfying $\inf\varphi>-\infty$, $\sup\varphi=\infty$ and $\mathrm{var}_1(\varphi)<\infty$. Let $\{\nu_n\}$ be a sequence in $\mathcal{M}$ such that $\nu_n\to\nu_\infty$ in the weak*-topology as $n\to\infty$. If $\int\varphi d\nu_\infty=\infty$ then $\int\varphi d\nu_n\to\infty$ as $n\to\infty$.*

*Proof.* Since $\sup\varphi=\infty$ and $\mathrm{var}_1(\varphi)<\infty$ it is possible to choose $k_0\geq 0$ such that $\inf\{\varphi(x)\colon x_0\geq k_0\}>0$. For each $k\geq k_0$ define $\varphi_k\colon X\to\mathbb{R}$ by $\varphi_k(x)=\varphi$ if $x_0\leq k-1$ and $\varphi_k(x)=0$ if $x_0\geq k$. Since $\varphi_k\leq\varphi_{k+1}$ and $\varphi_k\to\varphi$ pointwise as $k\to\infty$, the monotone convergence theorem gives $\int\varphi_k d\nu_\infty\to\int\varphi d\nu_\infty=\infty$. For each $L>0$ fix $k\geq k_0$ such that $\int\varphi_k d\nu_\infty\geq L$. Since $\varphi_k$ is bounded continuous, the weak*-convergence gives $\int\varphi_k d\nu_n\to\int\varphi_k d\nu_\infty$ as $n\to\infty$ and hence there exists $n_0\geq 0$ such that $\int\varphi_k d\nu_n\geq L/2$ for every $n\geq n_0$. Moreover, $\int\varphi d\nu_n\geq\int\varphi_k d\nu_n\geq L/2$ holds since $\varphi\geq\varphi_k$. It follows that $\int\varphi d\nu_n\to\infty$ as $n\to\infty$. □

**Lemma 2.3.** *Let $\{\nu_n\}$ be a sequence in $\mathcal{M}_\phi(\sigma)$ such that $F(\nu_n)$ converges to a finite number as $n\to\infty$. Then*

$$\sup_n\chi(\nu_n)<\infty.$$

*Proof.* For each $\alpha \geq \frac{\sqrt{5}+1}{2}$ put

$$b(\alpha) = \sup\left\{\frac{h(\mu)}{\chi(\mu)} \colon \mu \in \mathcal{M}(\sigma),\ \chi(\mu) = \alpha\right\}.$$

This number coincides with the Hausdorff dimension of the set of points having $\alpha$ as its Lyapunov exponent, see [27]. If the desired upper bound is false, then taking a subsequence if necessary we may assume $\chi(\nu_n) \to \infty$ as $n \to \infty$. Since $F(\nu_n)$ converges, $h(\nu_n)/\chi(\nu_n) \to 1$ and so $\lim_{\alpha\to\infty} b(\alpha) = 1$. This yields a contradiction to [17, Theorem 1.3] which asserts $\limsup_{\alpha\to\infty} b(\alpha) < 1$. □

2.3. **Identification of the convergence point.** Having shown that the convergence point belongs to $\mathcal{M}_\phi(\sigma)$, we next show that this point is $\mu_T$. To this end, we would like to use an upper semi-continuity argument. Unfortunately, neither the entropy or the (minus of the) free energy is upper semi-continuous at every measure which has finite entropy. However, entropy divided by Lyapunov exponent is upper semi-continuous by [10, lemma 6.5], and this suffices for our purpose.

**Lemma 2.4.** *Let $\{\nu_n\}$ be a sequence in $\mathcal{M}_\phi(\sigma)$ such that $F(\nu_n) \to 0$ as $n \to \infty$. If $\{\nu_n\}$ converges in the weak*-topology to a measure $\nu_\infty \in \mathcal{M}_\phi(\sigma)$, then $\nu_\infty = \mu_T$.*

*Proof.* Since $\inf_n \chi(\nu_n) > 0$ and $F(\nu_n) \to 0$,

$$0 = \lim_{n\to\infty} \frac{h(\nu_n)}{\chi(\nu_n)} - 1.$$

By [10, Lemma 6.5],

$$0 = \lim_{n\to\infty} \frac{h(\nu_n)}{\chi(\nu_n)} - 1 \leq \frac{h(\nu_\infty)}{\chi(\nu_\infty)} - 1.$$

Since $\chi(\nu_\infty) < \infty$, $F(\nu_\infty) \geq 0$ holds. The variational principle yields $\nu_\infty = \mu_T$. □

*Proof of the Main Theorem.* Theorem 1.1 implies $J(\alpha) = I(\alpha)$ for every $\alpha \in \mathbb{R}$. All that remains to show is $I(\alpha) > 0$ for every $\alpha > \alpha^-$ and $I(\alpha) \searrow 0$ as $\alpha \to \infty$.

*Proof of $I(\alpha) > 0$ for every $\alpha > \alpha^-$.* The assumption $\liminf_{n\to\infty} \psi(n) = \infty$ implies $\alpha^+ = \infty$. Hence, for every $\alpha > \alpha^-$ there exists a sequence $\{\mu_n\}$ in $\mathcal{M}_\phi(\sigma)$ such that $\psi \circ a_1 \in L^1(\mu_n)$ for every $n$ and $\int \psi \circ a_1 d\mu_n \to \alpha$. To conclude $I(\alpha) > 0$ it is enough to show that the sequence $\{F(\mu_n)\}_n$ does not accumulate on 0. Suppose this is false. Then taking a subsequence if necessary we may assume $F(\mu_n) \to 0$ as $n \to \infty$. Since $\liminf_{n\to\infty} \psi(n) = \infty$ and $\int \psi \circ a_1 d\mu_n \to \alpha$, $\{\mu_n\}$ is tight by Lemma 2.1. By Prohorov's theorem there exists a limit point, say $\mu_\infty \in \mathcal{M}$. Since $\mathcal{M}(\sigma)$ is weak*-closed, $\mu_\infty \in \mathcal{M}(\sigma)$ holds.

If $\chi(\mu_\infty) = \infty$ then $\chi(\mu_n) \to \infty$ by Lemma 2.2. This yields a contradiction to Lemma 2.3 and thus $\chi(\mu_\infty) < \infty$. From Lemma 2.4, $\mu_\infty = \mu_T$ holds. Since $\int \psi \circ a_1 d\mu_T = \infty$ from the assumption in the Main Theorem, Lemma 2.2 gives $\int \psi \circ a_1 d\mu_n \to \infty$ and a contradiction arises. Therefore we conclude $I(\alpha) > 0$.

*Proof of $I(\alpha) \searrow 0$ as $\alpha \to \infty$.* The upper bound in Theorem 1.1 with $C = [1, \infty)$ implies $\inf I|_{[1,\infty)} = 0$. Hence one must have $I(\alpha) \to 0$ as $\alpha \to \infty$. The strict monotonicity of $I$ follows from this and the convexity of $I$. $\square$

## 3. Level-1 LDP for arithmetic function

In this section we prove Theorem 1.1. The convexity and the lower semi-continuity of $I$ are straightforward from the definition. We show the lower and upper bounds, and then the last assertion on $I$.

*A word of length $n$* is an $n$-string of elements of $\mathbb{N}$. For two words $u = a_0 \cdots a_{m-1}$, $v = b_0 \cdots b_{n-1}$ denote by $uv$ the concatenated word $a_0 \cdots a_{m-1} b_0 \cdots b_{n-1}$ of length $m + n$. This notation extends in the obvious way to concatenations of arbitrary finite number of words. Denote by $E^n$ the set of words of length $n$. For each $w = a_0 \cdots a_{n-1} \in E^n$ define *a cylinder set of length $n$* by

$$[w] = [a_0, \ldots, a_{n-1}] = \{x \in X \colon x_i = a_i \text{ for every } i \in \{0, 1, \ldots, n-1\}\}.$$

There exist constants $c_0 > 0$, $c_1 > 0$ such that for every integer $n \geq 1$, every $w \in E^n$ and every $x \in [w]$,

$$c_0 \leq \frac{\lambda[w]}{e^{S_n \phi(x)}} \leq c_1. \tag{3.1}$$

3.1. **Lower bound.** A proof of the lower bound in Theorem 1.1 is based on the next result.

**Proposition 3.1.** *Let $\psi \colon \mathbb{N} \setminus \{0\} \to \mathbb{R}$. For every open interval $J$ and every $\mu \in \mathcal{M}_\phi^e(\sigma)$ for which $\int \psi \circ a_1 d\mu \in J$,*

$$\liminf_{n \to \infty} \frac{1}{n} \log \lambda_n(J) \geq F(\mu).$$

*Proof of the lower bound in Theorem 1.1.* Let $G \subset \mathbb{R}$ be an arbitrary open set. Since open intervals with rational endpoints form a countable base of topology of $\mathbb{R}$, Proposition 3.1 implies that for every $\mu \in \mathcal{M}_\phi^e(\sigma)$ with $\psi \circ a_1 \in L^1(\mu)$ and $\int \psi \circ a_1 d\mu \in G$,

$$\liminf_{n \to \infty} \frac{1}{n} \log \lambda_n(G) \geq F(\mu).$$

This yields

$$\liminf_{n \to \infty} \frac{1}{n} \log \lambda_n(G) \geq \sup \left\{ F(\mu) \colon \mu \in \mathcal{M}_\phi^e(\sigma), \int \psi \circ a_1 d\mu \in G \right\}.$$

For a non-ergodic $\mu \in \mathcal{M}_\phi(\sigma)$ with $\psi \circ a_1 \in L^1(\mu)$, by [15, Lemma 3.2] there exists a sequence $\{\mu_n\}$ in $\mathcal{M}_\phi^e(\sigma)$ such that $\psi \circ a_1 \in L^1(\mu_n)$ for every $n$ and $h(\mu_n) \to h(\mu)$, $\int \phi d\mu_n \to \int \phi d\mu$, $\int \psi \circ a_1 d\mu_n \to \int \psi \circ a_1 d\mu$ as $n \to \infty$. Hence the above inequality continues to hold even if $\mathcal{M}_\phi^e(\sigma)$ is replaced by $\mathcal{M}_\phi(\sigma)$. Using the lemma below yields the desired lower bound in Theorem 1.1.

**Lemma 3.2.** *For every set $A \subset \mathbb{R}$,*

$$\inf \left\{ -F(\mu) \colon \mu \in \mathcal{M}_\phi(\sigma), \int \psi \circ a_1 d\mu \in A \right\} \geq \inf I|_A,$$

*and the equality holds if $A$ is an open set.*

*Proof.* The definition of $I$ in Theorem 1.1 immediately yields

$$\inf\left\{-F(\mu)\colon \mu\in\mathcal{M}_\phi(\sigma), \int\psi\circ a_1 d\mu=\alpha\right\}\geq I(\alpha)$$

for every $\alpha\in\mathbb{R}$. Hence the desired inequality holds. If $A$ is a non-empty open set, then for each $\alpha\in A$ choosing $\epsilon>0$ such that $(\alpha-\epsilon,\alpha+\epsilon)\subset A$ we have

$$\begin{aligned}I(\alpha)&\geq\inf\left\{-F(\mu)\colon \mu\in\mathcal{M}_\phi(\sigma),\ \left|\int\psi\circ a_1 d\mu-\alpha\right|<\epsilon\right\}\\&\geq\inf\left\{-F(\mu)\colon \mu\in\mathcal{M}_\phi(\sigma),\ \int\psi\circ a_1 d\mu\in A\right\}.\end{aligned}$$

Taking the infimum over all $\alpha\in A$ yields the reverse inequality. □

For a proof of Proposition 3.1 we need the next lemma which permits us to approximate an ergodic measure with a finite collection of cylinder sets in a particular sense. Although this type of result is known in full generality (see e.g. [10, Proposition 3.1]), for completeness we include a proof adapted to our specific context.

**Lemma 3.3.** *Let $\mu\in\mathcal{M}_\phi^e(\sigma)$ and assume $\psi\circ a_1\in L^1(\mu)$. For every $\epsilon>0$ there exist an integer $k>1$ and a finite set $F^k\subset E^k$ such that*

$$\left|\frac{1}{k}\log\#F^k-h(\mu)\right|<\epsilon, \tag{3.2}$$

*and the following holds for every $w\in F^k$;*

$$\sup_{[w]}\left|\frac{1}{k}S_k\phi-\int\phi d\mu\right|<\epsilon; \tag{3.3}$$

$$\sup_{[w]}\left|\frac{1}{k}S_k(\psi\circ a_1)-\int\psi\circ a_1 d\mu\right|<\epsilon. \tag{3.4}$$

*Proof.* Put $\mathscr{A}=\{[i]\colon i\in\mathbb{N}\}$. Denote by $h(\mu,\mathscr{A})$ the entropy of $\mu$ with respect to $\sigma$ and the countably infinite partition $\mathscr{A}$. Since $h(\mu)<\infty$ and $\mathscr{A}$ is a generator, $-\sum_{i\in\mathbb{N}}\mu[i]\log\mu[i]<\infty$ and $h(\mu,\mathscr{A})=h(\mu)$ hold.

Let $\epsilon>0$. For each integer $k>1$ denote by $\mathscr{B}_k$ the set of $B\in\bigvee_{i=0}^{k-1}\sigma^{-i}\mathscr{A}$ such that

$$e^{-\left(h(\mu)+\frac{\epsilon}{2}\right)k}<\mu(B)<e^{-\left(h(\mu)-\frac{\epsilon}{2}\right)k}, \tag{3.5}$$

and the following holds for some $x\in B$:

$$\left|\frac{1}{k}S_k\phi(x)-\int\phi d\mu\right|<\frac{\epsilon}{2}; \tag{3.6}$$

$$\left|\frac{1}{k}S_k(\psi\circ a_1)(x)-\int\psi\circ a_1 d\mu\right|<\epsilon. \tag{3.7}$$

From Shannon-McMillan-Breiman's Theorem and Birkhoff's Ergodic Theorem, $\mu(\bigcup_{B\in\mathscr{B}_k} B)\to 1$ as $k\to\infty$. (3.5) implies

$$\frac{1}{2}e^{\left(h(\mu)-\frac{\epsilon}{2}\right)k}\leq \#\mathscr{B}_k\leq e^{\left(h(\mu)+\frac{\epsilon}{2}\right)k}.$$

Set $F^k=\{w\in E^k\colon [w]\in\mathscr{B}_k\}$. For $k$ large enough, we obtain (3.2).

Since $T$ satisfies Rényi's condition and $T^2$ is uniformly expanding [6, Chapter 4], $\sup_{w\in E^k}\sup_{x,y\in[w]}S_k\phi(x)-S_k\phi(y)$ is uniformly bounded in $k$. From this and (3.6) we obtain (3.3) for $k$ large enough. Since $\psi\circ a_1$ depends only on the first coordinate, $S_k(\psi\circ a_1)(x)=S_k(\psi\circ a_1)(y)$ holds for all $x,y\in[w]$ and every $w\in\mathscr{B}_k$. From this and (3.7) we obtain (3.4). □

*Proof of Proposition 3.1.* Since $J$ is open it is possible to choose $\epsilon>0$ such that $(\int\psi\circ a_1d\mu-2\epsilon,\int\psi\circ a_1d\mu+2\epsilon)\subset J$. For this $\epsilon$ fix an integer $k>1$ and a finite set $F^k\subset E^k$ for which the conclusions of Lemma 3.3 hold. For each $l\in\{1,\ldots,k-1\}$ denote by $P^l$ the set of $w\in E^l$ for which there exists $w\in E^{k-l}$ such that $w_*w\in F^k$. Put $P^0=\emptyset$.

Let $n\geq k$ be an integer and write $n=mk+l_0$ where $m,l_0$ are integers with $m\geq 1$ and $0\leq l_0<k$. Denote by $G^n$ the subset of $E^n$ which consists of words of the form $w_1w_2\cdots w_mw_*$ with $w_1,\ldots,w_m\in F^k$ and $w_*\in P^{l_0}$. Lemma 3.3 gives

$$\#G^n\geq(\#F^k)^m>e^{(h(\mu)-\epsilon)km}. \tag{3.8}$$

If $l_0=0$ then by Lemma 3.3 the following holds for every $w\in G^n$:

$$\inf S_n\phi|_{[w]}\geq\left(\int\phi d\mu-\epsilon\right)n\ \text{ and }\ \left|\frac{1}{n}S_n(\psi\circ a_1)(x)-\int\psi\circ a_1d\mu\right|<\epsilon. \tag{3.9}$$

For the rest of this paragraph we show that in the case $l_0\neq 0$ the two inequalities in (3.9) continue to hold with $\epsilon$ replaced by $2\epsilon$ and sufficiently large $n$. Since $\mathrm{var}_1(\phi)<\infty$ and $F^k$ is a finite set, $S_l\phi|_{\bigcup_{\omega\in P^l}[\omega]}$ is bounded for every $l\in\{1,\ldots,k-1\}$. For $n$ large enough and every $w\in G^n$ we have

$$\begin{aligned}\inf S_n\phi|_{[w]}&\geq\left(\int\phi d\mu-\epsilon\right)mk+\inf S_{l_0}\phi|_{\bigcup_{w\in P^{l_0}}[w]}\\&\geq\left(\int\phi d\mu-2\epsilon\right)n.\end{aligned}$$

Since $\psi\circ a_1$ depends only on the first coordinate and $F^k$ is a finite set, $S_l(\psi\circ a_1)|_{\bigcup_{\omega\in P^l}[\omega]}$ is bounded for every $l\in\{1,\ldots,k-1\}$. For $n$ large enough and every $w\in G^n$ we have

$$\begin{aligned}\inf S_n(\psi\circ a_1)|_{[w]}&\geq\left(\int\psi\circ a_1d\mu-\epsilon\right)mk+\inf S_{l_0}(\psi\circ a_1)|_{\bigcup_{w\in P^{l_0}}[w]}\\&\geq\left(\int\psi\circ a_1d\mu-2\epsilon\right)n\end{aligned}$$

and

$$\begin{aligned}\sup S_n(\psi\circ a_1)|_{[w]} &\leq \left(\int \psi\circ a_1 d\mu+\epsilon\right)mk+\sup S_{l_0}(\psi\circ a_1)|_{\bigcup_{w\in P^{l_0}}[w]}\\ &\leq \left(\int \psi\circ a_1 d\mu+2\epsilon\right)n.\end{aligned}$$

From (3.1) and the first inequality in (3.9),

$$\lambda[w]\geq c_0e^{\inf S_n\phi|_{[w]}}\geq c_0e^{(\int \phi d\mu-2\epsilon)n}\quad \text{for every } w\in G^n. \tag{3.10}$$

By the second inequality in (3.9),

$$\left\{x\in X\colon \frac{1}{n}S_n(\psi\circ a_1)\in J\right\}\supset \bigcup_{w\in G^n}[w].$$

From (3.8) and (3.10) we obtain

$$\begin{aligned}\frac{1}{n}\log \lambda_n(J)&\geq \frac{1}{n}\log\lambda\left(\bigcup_{w\in G^n}[w]\right)\\ &\geq \frac{1}{n}\log\left(\#G^n\inf_{w\in G^n}\lambda[w]\right)\\ &\geq F(\mu)-4\epsilon+\frac{1}{n}\log c_0.\end{aligned}$$

Letting $n\to\infty$ and then $\epsilon\to 0$ yields the desired inequality. □

3.2. **Upper bound.** We now show the upper bound (1.1) for every closed set.

**Proposition 3.4.** *Let $\psi\colon \mathbb{N}\setminus\{0\}\to\mathbb{R}$. For every interval $J$,*

$$\limsup_{n\to\infty}\frac{1}{n}\log\lambda_n(J)\leq -\inf I|_J.$$

*If $\lambda_n(J)\neq 0$ for infinitely many $n$, then* $\inf I|_J<\infty$.

*Proof of the upper bound in Theorem 1.1.* Let $C\subset\mathbb{R}$ be a closed set. First of all, assume $\inf I|_C<\infty$. Assume $C$ is bounded. Let $J_1,J_2,\ldots,J_p$ be a finite collection of intervals which altogether cover $C$. By Proposition 3.4,

$$\begin{aligned}\limsup_{n\to\infty}\frac{1}{n}\log\lambda_n(C)&\leq \limsup_{n\to\infty}\frac{1}{n}\log\lambda_n\left(\bigcup_{i=1}^p J_i\right)\\ &\leq -\inf_{i\in\{1,\ldots,p\}}\inf I|_{J_i}\\ &= -\inf I|_{\bigcup_{i=1}^p J_i}.\end{aligned}$$

Taking the infimum over all finite collections of intervals which altogether cover $C$,

$$\limsup_{n\to\infty}\frac{1}{n}\log\lambda_n(C)\leq \inf(-\inf I|_{\bigcup_{i=1}^p J_i})\leq -\inf I|_C, \tag{3.11}$$

where the last inequality holds because $I$ is lower semi-continuous, $C$ is compact and $\inf I|_C$ is attained.

Assume $C$ is unbounded. If $\inf I|_C$ is attained, then (3.11) remains to hold and the lower semi-continuity of $I$ yields the desired inequality. Assume $\inf I|_C$ is not attained. This implies that there is a sequence $\{x_n\}_n$ in $C$ such that $|x_n| \to \infty$ and $I(x_n) \to \inf I|_C$ as $n \to \infty$. Without loss of generality we may assume $x_n \to \infty$. Since $I$ is convex, $I(\alpha) \to \inf I$ as $\alpha \to \infty$ and $\inf I = \inf|_C$. Hence the desired inequality holds.

Next, assume $\inf I|_C = \infty$. The last assertion of Theorem 1.1 proved at the end of this paper gives $C \subset (-\infty, \alpha^-) \cup (\alpha^+, \infty)$. Take a subinterval $J^-$ (resp. $J^+$) of $(-\infty, \alpha^-)$ (resp. $(\alpha^+, \infty)$) containing $C \cap (-\infty, \alpha^-)$ (resp. $C \cap (\alpha^+, \infty)$). By the last assertion of Proposition 3.4, $\lambda_n(J^-) \neq 0$ only for finitely many $n$. Hence $\lambda_n(C \cap (-\infty, \alpha^-)) \neq 0$ only for finitely many $n$. In the same way, $\lambda_n(C \cap (\alpha^+, \infty)) \neq 0$ only for finitely many $n$. The desired inequality holds trivially: $-\infty \leq -\infty$. $\square$

For a proof of Proposition 3.4 we need the following result which can be proved along the well-known line of the thermodynamic formalism [4, 28].

**Lemma 3.5.** *Let $J$ be an interval. Let $n \geq 1$ be an integer and $D^n$ a finite subset of $E^n$ such that $\frac{1}{n} S_n(\psi \circ a_1)(x) \in J$ holds for every $x \in \bigcup_{w \in D^n} [w]$. There exists a measure $\mu \in \mathcal{M}(\sigma)$ supported on a compact set such that*

$$\log \sum_{w \in D^n} \lambda[w] \leq F(\mu) n + \log c_1 \quad \textit{and} \quad \int \psi \circ a_1 d\mu \in J.$$

*Proof.* Put $\widehat{\sigma} = \sigma^n$ and $\Lambda = \bigcap_{m=0}^{\infty} \widehat{\sigma}^{-m} \left( \bigcup_{w \in D^n} [w] \right)$. Then $\Lambda$ is a compact set and $\widehat{\sigma}|_\Lambda \colon \Lambda \to \Lambda$ is continuous. Put $\widehat{\phi} = S_n \phi$ and fix $y_0 \in \Lambda$. There exists a constant $c > 0$ such that $\sum_{i=0}^{m-1} (\widehat{\phi}(\widehat{\sigma}^i(x)) - \widehat{\phi}(\widehat{\sigma}^i(y))) \leq c$ for every $m \geq 1$, every $x, y \in \Lambda$ such that $\widehat{\sigma}^i(x), \widehat{\sigma}^i(y)$ belong to the same element of $D^n$ for every $i \in \{0, 1, \ldots, m-1\}$. By [4, Lemma 1.20],

$$\sup_{\widehat{\nu} \in \mathcal{M}(\widehat{\sigma}|_\Lambda)} \left( h_{\widehat{\sigma}|_\Lambda}(\widehat{\nu}) + \int \widehat{\phi} d\widehat{\nu} \right) = \lim_{m \to \infty} \frac{1}{m} \log \sum_{x \in (\widehat{\sigma}|_\Lambda)^{-m}(y_0)} e^{\sum_{i=0}^{m-1} \widehat{\phi}(\widehat{\sigma}^i(x))},$$

with $\mathcal{M}(\widehat{\sigma}|_\Lambda)$ the space of $\widehat{\sigma}|_\Lambda$-invariant Borel probability measures endowed with the weak*-topology and $h_{\widehat{\sigma}|_\Lambda}(\widehat{\nu})$ the entropy of $\widehat{\nu} \in \mathcal{M}(\widehat{\sigma}|_\Lambda)$ with respect to $\widehat{\sigma}|_\Lambda$. By (3.1), $e^{\widehat{\phi}(x)} \geq c_1^{-1} \lambda[w]$ holds for every $x \in [w]$ and every $w \in D^n$. Hence

$$\sum_{x \in (\widehat{\sigma}|_\Lambda)^{-m}(y_0)} e^{\sum_{i=0}^{m-1} \widehat{\phi}(\widehat{\sigma}^i(x))} \geq \left( \inf_{y' \in \Lambda} \sum_{x \in (\widehat{\sigma}|_\Lambda)^{-1}(y')} e^{\widehat{\phi}(x)} \right)^m \geq \left( c_1^{-1} \sum_{w \in D^n} \lambda[w] \right)^m.$$

Taking logs of both sides, dividing by $m$ and plugging the result into the previous inequality gives

$$\lim_{m \to \infty} \frac{1}{m} \log \sum_{x \in (\widehat{\sigma}|_\Lambda)^{-m}(y_0)} e^{\sum_{i=0}^{m-1} \widehat{\phi}(\widehat{\sigma}^i(x))} \geq \log \left( \sum_{w \in D^n} \lambda[w] \right) - \log c_1.$$

Plugging this into the previous inequality yields

$$\sup_{\widehat{\nu} \in \mathcal{M}(\widehat{\sigma}|_\Lambda)} \left( h_{\widehat{\sigma}|_\Lambda}(\widehat{\nu}) + \int \widehat{\phi} d\widehat{\nu} \right) \geq \log \left( \sum_{w \in D^n} \lambda[w] \right) - \log c_1.$$

Since $\mathcal{M}(\widehat{\sigma}|_\Lambda)$ is compact and $\mathcal{M}(\widehat{\sigma}|_\Lambda) \ni \widehat{\nu} \mapsto h_{\widehat{\sigma}|_\Lambda}(\widehat{\nu}) + \int \widehat{\phi} d\widehat{\nu}$ is upper semi-continuous, there exists a measure $\widehat{\mu} \in \mathcal{M}(\widehat{\sigma}|_\Lambda)$ which attains this supremum. The measure $\mu = \frac{1}{n}\sum_{k=0}^{n-1}(\sigma_*)^k(\widehat{\mu})$ is in $\mathcal{M}(\sigma)$ and satisfies the desired properties. □

*Proof of Proposition 3.4.* Let $J$ be an interval and $n_0 \geq 1$ an integer in Lemma 3.5. Put

$$H^n = \left\{ w \in E^n \colon [w] \cap \left\{ x \in (0,1) \colon \frac{1}{n} S_n(\psi \circ a_1) \in J \right\} \neq \emptyset \right\}.$$

If $H^n \neq \emptyset$ for only finitely many $n$, then the desired inequality holds trivially: $\infty \leq \infty$. Assume $H^n \neq \emptyset$ for infinitely many $n$. Fix such an $n$. Since $S_n(\psi \circ a_1)$ is constant on each cylinder set of length $n$, $\frac{1}{n}S_n(\psi \circ a_1)(x) \in J$ holds for every $x \in \bigcup_{w \in H^n}[w]$. Choose a finite subset $H^{n\prime}$ of $H^n$ such that

$$\log \sum_{w \in H^n} \lambda[w] \leq \log \sum_{w \in H^{n\prime}} \lambda[w] + 1.$$

By Lemma 3.5 there exists $\mu \in \mathcal{M}(\sigma)$ which is supported on a compact set and satisfies

$$\log \sum_{w \in H^{n\prime}} \lambda[w] \leq F(\mu)n + \log c_1 \quad \text{and} \quad \int \psi \circ a_1 d\mu \in J.$$

Since $\mu$ is supported on a compact set, $\mu \in \mathcal{M}_\phi(\sigma)$ holds. Therefore

$$\begin{aligned} \frac{1}{n}\log \lambda_n(J) &\leq F(\mu) + \frac{1}{n}(\log c_1 + 1) \\ &\leq \sup\left\{ F(\mu) \colon \mu \in \mathcal{M}_\phi(\sigma), \int \psi \circ a_1 d\mu \in J \right\} + \frac{1}{n}(\log c_1 + 1) \\ &\leq -\inf I|_J + \frac{1}{n}(\log c_1 + 1). \end{aligned}$$

The last inequality is by Lemma 3.2 and it implies the last assertion of Proposition 3.4. Letting $n \to \infty$ yields the desired inequality in Proposition 3.4. □

*Proof of the last assertion of Theorem 1.1.* Assume $\alpha^- < \alpha^+$. Clearly, $I(\alpha) < \infty$ for every $\alpha \in (\alpha^-, \alpha^+)$. Let $\alpha < \alpha^-$. If $I(\alpha) < \infty$ then there would exist $\mu \in \mathcal{M}_\phi(\sigma)$ for which $\int \psi \circ a_1 d\mu < \alpha$, a contradiction. In the same way, $I(\alpha) = \infty$ holds for every $\alpha > \alpha^+$. The finiteness of $I(\alpha^-)$, $I(\alpha^+)$ follows from the lower semi-continuity and the convexity of $I$. A slight modification of the argument covers the case $\alpha^- = \alpha^+$. □

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