Asymptotic behavior of the iterates of weakly almost periodic Markov operators and invariant densities

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Contents

Introduction

Dynamical system:
A measurable transformation T on a measurable sp. $(X, \mathscr{F})$ (i.e., $T^{-1}\mathscr{F} \subset \mathscr{F}$).

$(X, \mathscr{F}, \mu, T)$: a measure preserving system
$\overset{\text{def}}{\Longleftrightarrow}$ $\mu \circ T^{-1} = \mu$ (μ: T-invariant meausre).

Birkhoff (1931)

$(X, \mathscr{F}, \mu, T)$: an ergodic **probability preserving** system,
$\forall f : X \to \mathbb{R}$ with $\int_X f d\mu < \infty$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f \circ T^i(x) = \int_X f d\mu \qquad \mu\text{—a.e. } x \in X.$$

(T, μ): ergodic
$\overset{\text{def}}{\Longleftrightarrow}$ $\mu(A) = 0$ or $\mu(A^c) = 0$ if $T^{-1}A = A$.

Introduction

W: a (weakly) wandering set

$\overset{\text{def}}{\iff}$ $\mu(T^{-n}W \cap T^{-m}W) = 0$,

(resp. $\exists\{n_i\}_{i\geq 1}$ s.t. $\mu(T^{-n_k}W \cap T^{-n_l}W) = 0$).

(T,μ): conservative $\overset{\text{def}}{\iff}$ any wandering set, $\mu(W) = 0$.

Hopf (1937)

$(X, \mathscr{F}, \mu, T)$: a conservative and ergodic σ-**finite measure preserving** system, $\forall f, g \in L^1$ with $\int_X g d\mu \neq 0$,

$$\lim_{n\to\infty} \frac{\sum_{i=0}^{n-1} f \circ T^i(x)}{\sum_{i=0}^{n-1} g \circ T^i(x)} = \frac{\int_X f d\mu}{\int_X g d\mu} \qquad \mu\text{–a.e. } x \in X.$$

Target: **an absolutely continuous** (σ-)**finite invariant measure** with respect to some reference measure.

Background

[Hajian, Kakutani 1964],[Sucheston 1964]
T: a nonsingular transformation on a prob. sp. $(X, \mathscr{F}, m)$ (i.e., $m(T^{-1}A) = 0$ if $m(A) = 0$).
The followings are equivalent:

- There exists an **equivalent** finite invariant measure;
- $\liminf_{n\to\infty} m \circ T^{-n}(A) > 0$ if $m(A) > 0$;
- $\liminf_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} m \circ T^{-i}(A) > 0$ if $m(A) > 0$;
- $\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} m \circ T^{-i}(A) > 0$ if $m(A) > 0$;
- There exists no weakly wandering set of positive m-measure.

Background

[Dean, Sucheston 1966]
$(X, \mathscr{F}, m)$: a probability sp.
P: a positive linear op. over $L^1(m)$ with $\|P\|_{\mathrm{op}} \leq 1$.
The followings are equivalen:

- There exists a **strictly positive** $f_0 \in L^1$ s.t. $Pf_0 = f_0$;
- $\inf_{n \geq 0} \int_A P^n 1_X dm$ if $m(A) > 0$;
- $\lim_{n\to\infty} \left[\sup_{j\geq 0} \frac{1}{n} \sum_{i=j}^{n+j-1} \int_A P^i 1_X dm \right] > 0$ if $m(A) > 0$;
- $\liminf_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \int_A P^i 1_X dm > 0$ if $m(A) > 0$.

Preliminaries

$(X, \mathscr{F}, m)$: a probability space,
L^1: all real-valued integrable functions over $(X, \mathscr{F}, m)$,
L^∞: all real-valued essentially bounded functions.

The Perron-Frobenius operator

$T : X \to X$ a measurable and nonsingular transformation.
The **Perron-Frobenius operator** corresponding to T:
$P : L^1 \to L^1$,

$$\int_A Pf dm = \int_{T^{-1}A} f dm \quad (f \in L^1, A \in \mathscr{F}).$$

The **Koopman operator** : $P^* : L^\infty \to L^\infty$, $P^* g = g \circ T$,

$$\int_X Pf \cdot g dm = \int_X f \cdot P^* g dm \quad (f \in L^1,\ g \in L^\infty).$$

Preliminaries

$(X, \mathscr{F}, m)$: a probability space.

Defnition (Markov operators)

A linear operator $P : L^1 \to L^1$ is a **Markov operator**.
$\overset{\text{def}}{\Longleftrightarrow} Pf \geq 0$ and $\|Pf\|_1 = \|f\|_1$ if $f \geq 0$.

Remark

- The Perron-Frobenius operator corresponding to a nonsingular transformation is also a Markov operator.
- $P^* : L^\infty \to L^\infty$ the adjoint op. $(P^* 1_X = 1_X)$
$$\int_X Pf \cdot g dm = \int_X f \cdot P^* g dm \ (f \in L^1,\ g \in L^\infty).$$
- $\mu \circ T^{-1} = \mu$ with $\mu \ll m \Leftrightarrow P \dfrac{d\mu}{dm} = \dfrac{d\mu}{dm}$.

Markov Processes

P: a Markov operator given.
$\rightsquigarrow$ The transition probability of the Markov process:

$$P(x, A) \coloneqq P^* 1_A(x) \qquad (\forall A \in \mathscr{F}).$$

Conversely,

$P(x, A)$: a transition probability of a Markov process given.
$\rightsquigarrow$ The Markov operator:

$$Pf \coloneqq \frac{d\left(\int_X f(x) P(x, \cdot) dm(x)\right)}{dm} \qquad (\forall f \in L^1).$$

Example (Additive Noise)

T: a measurable transformation on $([0,1], \mathscr{B}, \lambda)$,
the process: $x_{n+1} = T(x_n) + \xi_n \pmod 1$
$\xi_0, \xi_1, \cdots$: i.i.d. with the density k.
f_n: the distribution of x_n.

$$f_{n+1}(x) = \int_{[0,1]} f_n(y)k(x - T(y))dy.$$

The Markov operator P arising from this stochastic process is given by

$$Pf = \int_{[0,1]} f(y)k(x - T(y))dy.$$

Iwata, Ogihara (2013)

$\exists f_0 \in L^1_+$ with $\int_X f_0 dm = 1$ s.t. $Pf_0 = f_0$,

$$\lim_{n\to\infty} \|P^n f - f_0\|_1 = 0 \quad (\forall f \in L^1_+ \text{ with } \int_X f dm = 1).$$

The Existence of Absolutely Continuous Invariant Probabilities

Theorem (1)

$(X, \mathscr{F}, m)$: a probability space,
$T : X \to X$ a measurable and nonsingular transformation.
Then the followings are equivalent.

1 $\exists \mu \ll m$: a finite T-invariant measure s.t.

$$\bigcup_{n\geq 0} T^{-n}\left[\frac{d\mu}{dm} > 0\right] = X \mod m;$$

2 $\forall \epsilon > 0,\ \exists \delta > 0$, s.t.

$$[m(A) < \delta \Rightarrow \sup_{n\geq 0} m(T^{-n}A) < \epsilon]$$

(The unif. integrability of $\{\frac{d(m\circ T^{-n})}{dm}\}_{n\geq 0}$);

3 $\forall A \in \mathscr{F},\ \exists \lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} m\circ T^{-i}(A)$.

The Existence of Finite Invariant Densities for Markov Operators

Theorem (2)

$(X, \mathscr{F}, m)$: a probability space, $P : L^1 \to L^1$: a Markov operator. Then the followings are equivalent.

1. $\exists f_0 \in L^1_+ := \{f \in L^1 \mid f \geq 0\}$ s.t.
   - $Pf_0 = f_0$ m-a.e. and
   - $\lim_{n\to\infty} P^{*n} 1_{[f_0>0]} = 1_X$, $\qquad (P^{*n}1_{[f_0>0]} = 1_{T^{-n}[f_0>0]})$;
2. $\{P^n 1_X\}_n$: weakly precompact;
3. P: weakly almost periodic.

- $\{P^n f\}_n$: weakly precompact.
  $\overset{\text{def}}{\iff} \exists \{n_k\}_k$ s.t. $\exists \text{w-}\lim_{k\to\infty} P^{n_k} f$.
- P: weakly almost periodic.
  $\overset{\text{def}}{\iff} \forall f \in L^1$, $\{P^n f\}_n$: weakly precompact.

Conservative (Dissipative) Part for Transformations

T: a nonsingular transformation on $(X, \mathscr{F}, m)$.

Defnition (The conservative and dissipative part)

$\mathcal{W}$: the family of all wandering sets.

The **dissipative part**: $\mathfrak{D} := \bigcup_{W \in \mathcal{W}} W$.

The **conservative part**: $\mathfrak{C} := X \setminus \mathfrak{D}$.

Aaronson (1997)

$\forall u \in L^1$ with $u > 0$,

$$\mathfrak{C} = \left\{ x \in X \mid \sum_{n=0}^{\infty} P^n u(x) = \infty \right\}.$$

Conservative (Dissipative) Part for Operators

P: a Markov operator. $\forall u \in L^1$ with $u > 0$,

Defnition (The conservative and dissipative part)

The set (which is independent of the choice of u)

$$\mathfrak{C} := \left\{ x \in X \mid \sum_{n=0}^{\infty} P^n u(x) = \infty \right\}$$

is called the **conservative part** and

$$\mathfrak{D} := X \setminus \mathfrak{C}$$

is called the **dissipative part**.

The Induced Transformation

$T : X \to X$ a nonsingular transformation,

$$\exists E \in \mathscr{F} \text{ s.t. } \bigcup_{n \geq 1} T^{-n}E = X \mod m.$$

Then, we can define the *hitting time* of E for a.e. $x \in X$

$$\varphi_E(x) = \min\{n \geq 1 \mid x \in T^{-n}E\}.$$

The induced transformation

$$\begin{array}{rcl} T_E : X & \longrightarrow & E \\ \Psi & & \Psi \\ x & \longmapsto & T^{\varphi_E(x)}x \end{array}$$

is called the **induced transformation** on E.

Remark

The customary def. of the induced trans.: $T_E |_E : E \to E$.

The Induced Transformation

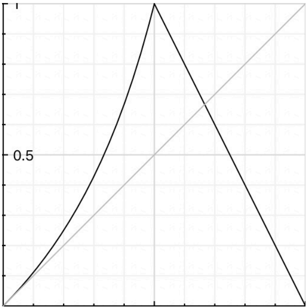

Figure: Graph of T

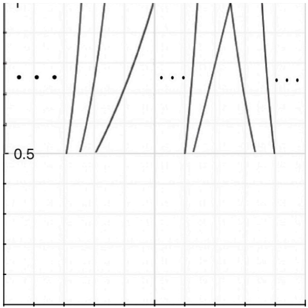

Figure: Graph of T_E

$$Tx = \begin{cases} \frac{x}{1-x} & x \in [0, 1/2) \\ -2x + 2 & x \in [1/2, 1] (=: E). \end{cases}$$

The Induced Operator

The induced operator

P: a Markov operator. Assume $\lim_{n\to\infty}(P^* I_{E^c})^n 1_X = 0$.
P_E: the **induced operator** on E,

$$P_E := (I_E P)\sum_{n=0}^{\infty}(I_{E^c}P)^n$$

(I_E: the restriction operator on E: $I_E f = f \cdot 1_E$).

$$P_E f = \sum_{n=0}^{\infty} P\underbrace{(P(\ldots P(Pf\cdot 1_{E^c})1_{E^c}\ldots)1_{E^c})}_{n-\text{th time}}1_E \quad (f \in L^1).$$

Remark

T: a nonsingular transformation
$\Rightarrow$ P_E: the Perron-Frobenius operator corresponding to T_E.

The Existence of Absolutely Continuous σ-finite Invariant Measures

Theorem (3)

$(X, \mathscr{F}, m)$: a probability space,
$T : X \to X$ a measurable and nonsingular transformation.
Then the followings are equivalent.

1. $\exists \mu \ll m$: a σ-finite T-invariant measure, $\exists A \subset [\frac{d\mu}{dm} > 0] \cap \mathfrak{C}$ with $\mu(A) < \infty$ s.t.
$$\bigcup_{n \geq 1} T^{-n} A = X \mod m;$$
2. $\exists E \in \mathscr{F}$ s.t. T_E is well-defined and $\exists \mu_E \ll m$: a finite T_E-invariant measure s.t.
$$\bigcup_{n \geq 1} T^{-n} \left[\frac{d\mu_E}{dm} > 0 \right] = X \mod m.$$

The Existence of σ-finite Invariant Densities for Markov Operators

Theorem (4)

$(X, \mathscr{F}, m)$: a probability space, P: a Markov operator.
Then the followings are equivalent.

1. $\exists h$: a non-negative measurable func.
   s.t. $\int_{\cdot} hdm$: σ-finite and $Ph = h$,
   $\exists A \subset [h > 0] \cap \mathfrak{C}$ with $\int_A hdm < \infty$ s.t.
   $$\lim_{n\to\infty} (P^* I_{A^c})^n 1_X = 0;$$
2. $\exists E \in \mathscr{F}$ s.t. P_E is well-defined and
   $\exists h^* \in L^1_+$ s.t. $P_E h^* = h^*$ and $\lim_{n\to\infty} (P^* I_{[h^*=0]})^n 1_X = 0$.

Future work
(Position Dependent Random Maps)

$(X, \mathscr{F}, m)$: a state space, $(W, \mathscr{B}, \nu)$: a parameter space,
$T_w : X \to X$ $(w \in W)$ a nonsingular transformation,
$p : W \times X \to [0, \infty)$ a probability density
(i.e., $\int_W p(w, x) d\nu(w) = 1$ for $x \in X$).
The **position dependent random map** is defined as a Markov process with the transition function

$$P(x, A) = \int_W p(w, x) 1_A(T_w(x)) d\nu(w).$$

[Inoue, 2012] gives a sufficient condition for the existence of an invariant density for P.

Target: a σ-finite infinite invariant measure.

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