Random holomorphic dynamics of Markov systems

Takayuki Watanabe

Kyoto University
joint work with: Hiroki Sumi (Kyoto University)

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Random (holomorphic) dynamical systems

Y : compact metric space
$y_0 \in Y$: initial point

<u>How do RANDOM orbits behave?</u>

$$y_0 \overset{f_1}{\mapsto} y_1 \overset{f_2}{\mapsto} y_2 \overset{f_3}{\mapsto} \cdots ?$$

where $f_1, f_2, f_3, \ldots$ are randomly chosen. In this talk, we consider the random dynamics on the Riemann sphere $\hat{\mathbb{C}}$ whose choices of maps are not independent and identically distributed but obey "Markovian rules".

Motivative example

- $\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\} \overset{top}{\simeq} S^2$: the Riemann sphere
- $f_1, \ldots, f_m$: polynomial maps on $\hat{\mathbb{C}}$

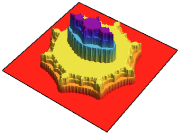
Definition of random orbits

0 Fix an initial point $z_0 \in \hat{\mathbb{C}}$.

1 Choose a polynomial f_{ω_1} with probability p_{ω_1} and define $z_1 = f_{\omega_1}(z_0)$.

n After choosing $f_{\omega_{n-1}}$, choose a polynomial f_{ω_n} with probability $p_{\omega_{n-1}\omega_n}$ and define $z_n = f_{\omega_n}(z_{n-1})$ for each step.

We are especially intersted in the probablity of random orbits tending to ∞ and the chaotic initial points (or the Julia set).



1 Settings

2 Main results (random polynomial dynamics)

Definition of τ

Let $m \in \mathbb{N}$ and let τ_{ij} be a Borel measure on the space $\mathrm{OCM}(Y)$ of all open continuous maps on Y for each $1 \leq i, j \leq m$. Set $p_{ij} := \tau_{ij}(\mathrm{OCM}(Y))$ and suppose that $\sum_{j=1}^{m} p_{ij} = 1$ for all $i = 1, \ldots, m$.

Purpose

We want to investigate the Markov chain on $Y \times \{1, \ldots, m\}$ with transition probability $\mathbb{P}((y, i), B \times \{j\}) = \tau_{ij}(\{f \in \mathrm{OCM}(Y); f(y) \in B\})$ from a point $(y, i) \in Y \times \{1, \ldots, m\}$ to a Borel set $B \times \{j\}$.

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Remark 1

If $\underline{m = 1}$, this Markov chain is i.i.d. random dynamical system.

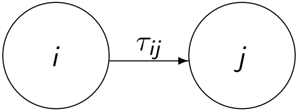
We always assume that the matrix $P = (p_{ij})$ is irreducible, i.e. $\forall i, j = 1, \ldots, m$, $\exists N \in \mathbb{N}$ such that the (i,j)-component of P^N is positive.

Definition of Markov systems

For this family $\tau = (\tau_{ij})_{i,j=1,\dots,m}$ of measures, we define the "Markov system S_τ" in the following way.

1. The vertex set $V := \{1, 2, \cdots, m\}$.
2. The edge set $E := \{(i,j) \in V \times V;\ p_{ij} > 0\}$.

We regard (V, E) as a directed graph. For all $e \in E$, we denote $e = (i(e), t(e))$ and we call i(e) (resp. t(e)) the initial (resp. terminal) vertex. We call $S_\tau := (V, E, (\operatorname{supp} \tau_e)_{e \in E})$ the Markov system induced by τ.

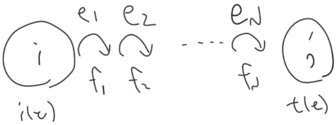


Definition of Julia sets

Let $S_\tau := (V, E, (\operatorname{supp}\tau_e)_{e\in E})$ be a Markov system induced by τ.

1. A word $e = (e_1, \ldots, e_N) \in E^N$ with length $N \in \mathbb{N}$ is said to be admissible if $t(e_n) = i(e_{n+1})$ for all $n = 1, 2, \cdots, N-1$. For this word e, we call $i(e_1)$ (resp. $t(e_N)$) the initial (resp. terminal) vertex of e and we denote it by $i(e)$ (resp. $t(e)$).
2. For all $i, j \in V$, we set

$$H_i^j(S_\tau) := \{f_N \circ \cdots \circ f_1;\, f_n \in \operatorname{supp}\tau_{e_n},\, i = i(e_1),\, t(e_N) = j,\\ (e_1, \ldots, e_N) \text{ is an admissible word with length } N\}.$$



Definition of Julia sets

3. For each $i \in V$, we denote by $F_i(S_\tau)$ the set of all points $y \in Y$ for which there exists a neighborhood U in Y such that the family $\bigcup_{j \in V} H_i^j(S_\tau)$ is equicontinuous on U. $F_i(S_\tau)$ is called the Fatou set of S_τ at the vertex i and the complement $J_i(S_\tau) := Y \setminus F_i(S_\tau)$ is called the Julia set of S_τ at the vertex i.
4. The set $J_{\ker,i}(S_\tau) := \bigcap_{j \in V} \bigcap_{h \in H_i^j(S_\tau)} h^{-1}(J_j(S_\tau))$ is called the kernel Julia set of S_τ at the vertex $i \in V$.

Basic properties

1. If $m = 1$, the Julia set $J_1(S_\tau)$ is equal to the set of all points $y \in Y$ where the **semigroup** $H_1^1(S_\tau)$ is not equicontinuous on any neighborhood U of y in Y. (This is called the Julia set of the **semigroup** $H_1^1(S_\tau)$.)
2. The Fatou sets $F_i(S_\tau)$ is open subset of Y and the Julia set $J_i(S_\tau)$ is compact subset of Y for all $i \in V$.
3. $h(F_{i(e)}(S_\tau)) \subset F_{t(e)}(S_\tau)$, $h^{-1}(J_{t(e)}(S_\tau)) \subset J_{i(e)}(S_\tau)$ for all $h \in H_i^j(S_\tau)$.
4. Suppose that Y is locally connected and

$$\sup\{\operatorname{diam} B;\ \text{B is a connected component of } h^{-1}(B(y,\varepsilon))\} \to 0$$

as $\varepsilon \to 0$ for all point $y \in Y$ and for all $h \in H_i^j(S_\tau)$. Then $J_i(S_\tau)$ is equal to the Julia set of the semigroup $H_i^i(S_\tau)$ for all $i \in V$.

Definition of $\tilde{\tau}_i$ and a proposition

We define the Borel probability measures $\tilde{\tau}_i$ on $(\mathrm{OCM}(Y) \times E)^{\mathbb{N}}$ for $i \in V$, as follows. For N Borel sets $A_n\,(n = 1, \cdots, N)$ of $\mathrm{OCM}(Y)$ and for $(e_1, \ldots, e_N) \in E^N$, set $A'_n = A_n \times \{e_n\}$. We define the measure $\tilde{\tau}_i$ on $(\mathrm{OCM}(Y) \times E)^{\mathbb{N}}$ so that

$$\tilde{\tau}_i(A'_1 \times \cdots \times A'_N \times \prod_{N+1}^{\infty} (\mathrm{OCM}(Y) \times E))$$

$$= \begin{cases} \tau_{e_1}(A_1) \cdots \tau_{e_N}(A_N) & \text{, if } (e_1, \ldots, e_N) \text{ is admissible with } i(e_1) = i \\ 0 & \text{, otherwise.} \end{cases}$$

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Proposition 2

Let λ be a Borel finite measure on Y. Suppose that $J_{\mathrm{ker},j}(S_\tau) = \emptyset$ for some $j \in V$. Then,

$$\lambda(\{y \in Y; \{f_N \circ \cdots \circ f_1\}_{N \in \mathbb{N}} \text{ is not equiconti on any nbhd } U\}) = 0$$

for $\tilde{\tau}_i$ -a.e. $(f_n, e_n)_{n \in \mathbb{N}} \in (\mathrm{OCM}(Y) \times E)^{\mathbb{N}}$ and for all $i \in V$.

Polynomial maps and the probability of tending to ∞

In the following, we assume that

1. Y is the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\} \overset{top.}{\simeq} S^2$.
2. For all $e \in E$, $\operatorname{supp} \tau_e$ are compact subsets of the space Poly of all polynomial maps of degree 2 or more.

Note that ∞ is a common attracting fixed point of all polynomials $f\colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$.

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Definition 3

We define $\mathbb{T}_\infty : \hat{\mathbb{C}} \times \{1, \ldots, m\} \to [0, 1]$ by

$$\mathbb{T}_\infty(z, i) := \tilde{\tau}_i(\{(f_n, e_n)_{n\in\mathbb{N}} \in (\mathrm{Poly} \times E)^{\mathbb{N}}; \\ f_N \circ \cdots \circ f_1(z) \to \infty (N \to \infty)\}).$$

$\mathbb{T}_\infty$ represents the probability of tending to ∞.

Main results (1)

Main Result A

If $J_{\mathrm{ker},j}(S_\tau) = \emptyset$ for some $j \in V$, then $\mathbb{T}_\infty$ is continuous on $\hat{\mathbb{C}} \times \{1, \ldots, m\}$.

Main Result B

Suppose that there exists $e \in E$ such that

$$\mathrm{supp}\, \tau_e \supset \{f + c; |c - c_0| < \epsilon\}$$

for some $f \in \mathrm{Poly}, c_0 \in \mathbb{C}, \epsilon > 0$. Then $J_{\mathrm{ker},j}(S_\tau) = \emptyset$ for some $j \in V$.

Main results (2)

Main Result C

Suppose that

- $\operatorname{supp}\tau_e$ are finite set for all $e \in E$.
- For all $e_1, e_2 \in E$ with $i(e_1) = i(e_2)$ and for all $f_1 \in \operatorname{supp}\tau_{e_1}$, $f_2 \in \operatorname{supp}\tau_{e_2}$, we have $f_1^{-1}(J_{t(e_1)}(S_\tau)) \cap f_2^{-1}(J_{t(e_2)}(S_\tau)) = \emptyset$, except the case $e_1 = e_2$ and $f_1 = f_2$.

Then $\mathbb{T}_\infty \equiv 1$ or

$$J_i(S_\tau) = \{z \in \mathbb{C}; \mathbb{T}_\infty(\cdot, i) \text{ is not constant in any nbhd of } z\}$$

for all $i \in V$.

Main results (3)

Main Result D (randomness-induced phenomenon)

In addition to the assumption of Main Result C, if there exist $e_1, e_2 \in E$ such that $i(e_1) = i(e_2)$ and $e_1 \neq e_2$, then $\mathbb{T}_\infty$ is continuous on the whole space.

Note that

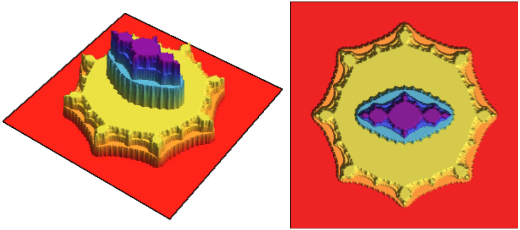
1. On the deterministic polynomial dynamics, $\mathbb{T}_\infty$ cannot be continuous on the whole space.
2. For $p = (p_1, \ldots, p_m)$ with $\sum_{i \in V} p_i = 1$, define $T_\infty \colon \hat{\mathbb{C}} \to [0, 1]$ by
$$T_\infty(z) := \sum_{i \in V} p_i \mathbb{T}_\infty(z, i).$$
If $m = 1$ (i.i.d. case), either $T_\infty \equiv 1$ or $\exists z_0 \in \hat{\mathbb{C}}$ s.t. $T_\infty(z_0) = 0$ [Sumi, 2011]. However, if $m \geq 2$ (non-i.i.d. case), there exists Markov systems S_τ, which satisfy the assumptions of Main Result D, such that $T_\infty \not\equiv 1$ and $\forall z \in \hat{\mathbb{C}},\ T_\infty(z) > 0$.

Example

Let $g_1(z) = z^2 - 1, g_2(z) = z^2/4$ and set $m = 2$,

$$(p_1, p_2) = (\frac{2}{3}, \frac{1}{3}), \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{pmatrix} \text{ and } f_i = g_i \circ g_i$$

for $i = 1, 2$. Define τ_{ij} as the Dirac measure $p_{ij}\delta_{f_i}$.



The graph of $1 - T_\infty$, which represents the probability of NOT tending to ∞, is continuous on $\hat{\mathbb{C}}$ and varies precisely on the Julia sets $\bigcup_{i \in V} J_i(S_\tau)$.

References

general theory (text book)

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holomorphic case

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3. H. Sumi, *"Random complex dynamics and semigroups of holomorphic maps"*, Proc. Lond. Math. Soc. (3) 102 (2011), no. 1, 50-112.
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Thank you!